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THE HISTORY OF SCIENCE.

INTRODUCTION.

[Dr. George Sarton is a Belgian scholar who has done much to promote the idea of a "History of Science" (as opposed to the history of any particular science, or to the sum of such particular histories.) He advocates a synthetic study that necessitates the collaboration of the scientist, the philosopher and the historian.

In 1913, Dr. George Sarton founded *Isis*, an international quarterly devoted to the history and to the organization of science, printed and published in Belgium. He himself lived a very quiet and retired life with his wife and daughter in his country home of Wondelgem, near Ghent—devoting all his time and a great deal of money to his historical studies. When the German invasion broke over Belgium, their income being entirely cut off, they had to leave their home; and after having buried all manuscripts in their garden, they went in a peasant cart to Holland, thence to England, and lastly came to this country. Dr. Sarton's library—one of the most complete on the subject he is studying—had to be abandoned: we sincerely hope that it will be saved and that Dr. Sarton will recover it after the war. He lectured in 1915 on the history of science at the summer school of the University of Illinois, at the George Washington University of Washington, and at Clark University. He has now been appointed lecturer at Harvard.

Dr. Sarton will resume the publication of *Isis* as soon as circumstances permit.—EDITOR.]

THIS essay is to explain the meaning of the history of science, to determine its limits and to show how it should be studied.

The history of science is the study of the development of science—just as one studies the development of a plant or an animal—from its very birth. We try to see it grow and unfold itself under many diverse conditions. And it is not enough—as we shall see further on—to study sep-

arately the development of each science; one has to study the development of all the sciences together. Besides, it is impossible to separate them satisfactorily one from the other; they grow together and mingle continually in innumerable ways.

There has been much research concerning the history of some particular sciences, and there are, for instance, excellent textbooks on the history of mathematics and of medicine, but there does not exist at the present day even a tolerably good history of science. The reader very likely knows the *History of the Inductive Sciences* by William Whewell, published in 1837. It was certainly a valuable book seventy years ago, but is now antiquated, and any one who does not know the history of science will do better not to use it at all. The best book that we have now at our disposal is that of Friedrich Dannemann,¹ but it is very elementary and can only be considered as a first and rough approximation. A bulky work published by Henry Smith Williams seems to be very popular in this country; at least, I have found copies of it in all the libraries where I have been. They are generally placed in the reference room where they are likely to be very often consulted. Owing to this, I feel obliged to say that these books are nothing but newspaper work, and quite unreliable.

While numberless books, many of them excellent, are published every year on the history of literature, of art, of religions, how is it that there is not yet a single history of science that can be compared with the best of them? When so many institutions, libraries, lectureships have been dedicated to the history of everything, how is it that the history of science has been so much neglected? The history of everything has been studied, except of that which

¹ Friedrich Dannemann, *Die Naturwissenschaften in ihrer Entwicklung und in ihrem Zusammenhange*. 4 vols., 1910-1913. I have analyzed this work at some length in *Isis*, II, pp. 218-222.

is the most distinctive feature of our civilization. How is that?

The most obvious, if not the best reasons, are the following. The people who have no knowledge of science, or but slight, are afraid of it. They are not inclined to read a book dealing with the history of science, because they think that they are not equal to appreciating it. Now this is a mistake: every intelligent man or woman can understand the development of science, at least if it be properly presented and taken from the beginning. More than that, I am convinced that the historical method is the best to convey scientific facts and ideas to unprepared minds and to make them thoroughly understandable,—at least that is so in the case of grown-up people. On the other hand, those who know science—or who are supposed to know it because they have made a special study in some narrow field—are often given to viewing history with contempt. They think that it is hopelessly inaccurate and, according to their own definition of science, unscientific. This is another mistake, which, however, it would take too long to completely refute. Suffice it to say that historical studies, like all other studies, are approximate; the approximation obtained by historians may be looser, but the studies are none the less scientific for that. It is not so much its degree of approximation, as a definite knowledge of this degree that gives to a study its scientific character.

At any rate, these reasons are only the most superficial ones. To set forth the others, I am obliged to make a short philosophical digression.

SCIENCE AND PHILOSOPHY.

Indeed, to make the real significance of our studies clear, it is necessary to impress the reader with a sense of the intellectual needs they must satisfy.

New scientific facts are discovered every day all over

the world and they continually make it necessary to revise our theories or to invent new ones. At the same time, science as a whole becomes more complete and deeper. Since the last century, its complexity has been developed to such a degree that now one of the first conditions of really original work is that it should be sufficiently specialized. The necessity of separating the difficulties in order the better to solve them, has made it more and more necessary to divide scientific work, and this division of labor seems to have reached a climax. That this tendency, which we may call the analytical tendency, has been extremely useful, the whole fabric of modern science is there to testify. However, its exclusive predominance is not without danger. This was not palpable in the beginning, but we see it clearly now. Indeed, the object of science is not to discover insulated facts, but to coordinate and to explain them one by the other. By dint of specialization, science would run the risk of missing its very aim; the quantity of scientific knowledge would increase, but it would be all in vain, the scientific spirit would be impoverished.

Besides, excessive analytical tendencies, without any counterpoise, would bring about another and a still graver danger: not only science would be menaced by disintegration, but our social life itself. Instead of bringing their fellow men together by giving them some common points of view, the scientists would finally be unable to understand one another.

This essential rhythm of our mind that makes us feel by turns the need of analysis and the need of synthesis, we find also in the changing idea that men have of the relations between science and philosophy. Indeed, there corresponds to it a similar rhythm which by turns brings together or drives asunder the scientist and the philosopher. A comparative study of the history of science and

of the history of philosophy would give us many opportunities to verify this.

The scientists of genius—I so call scientists whose discoveries revolutionize all accepted ideas and who originate studies of a radically new kind—have always exerted a considerable influence upon the evolution of philosophy. On the other hand, their own minds must have been of a very synthetical nature, and they have certainly borrowed much in a more or less conscious way from the philosophical store to formulate their revolutionary ideas. Think of Galilei, of Kepler, of Newton, of Darwin. Their work and influence cannot be understood, unless one takes into account these continuous interchanges between science and philosophy. They have drawn the desire of creating a new synthesis from the ideology of their time; and on the other hand, it is because their discoveries have deeply transformed this ideology that their influence has extended far beyond the scientific field where it originated.

In the same way the great philosophers—those who have really renewed the thought of their age—have also considerably influenced the progress of science. They were not themselves creative scientists, but at least they possessed all the scientific knowledge available to them. Think of Plato, Aristotle, Descartes, Leibniz, Kant. Here again, it is indispensable to conceive a double stream of ideas between science and philosophy. It is in the scientific domain that they have found at the same time the intuition of and the materials necessary to a new system; and this system in its turn, has renovated the philosophical atmosphere in which science was to pursue its development.

Therefore, those who study the history of philosophy ought to know the history of science. This is for the philosopher a heavy task, but I do not see how he can possibly escape from it. If one confines oneself to the study of, let us say, Descartes's philosophy, regardless of its conse-

quences in the field of mathematics, mechanics, astronomy, physics, medicine, botany, it stands to reason that it is impossible to give a complete or even a fair idea of his genius. Moreover, it is necessary to study the influence exerted by the Cartesian philosophy over the whole scientific thought of the seventeenth and eighteenth centuries, and even over our own science, and it is only in this way that Descartes's personality appears in its true light.

Everybody remembers those great epochs of synthesis of which Greek antiquity has given us some glorious examples, and nearer to us, the Renaissance and Cartesianism. On the contrary, during the nineteenth century, the analytical tendencies have been predominant. Synthetic construction sank into disrepute, partly as a result of the immense success of the inductive sciences, partly because most people were sick of the loose literature of the metaphysicians who came after Kant.

Whatever the case may be, a philosophical reaction was unavoidable, and this reaction still holds good, our own studies being only one aspect of it among many others. This reaction dates from the beginning of our century; it was in a great measure caused by the resounding discoveries of the last twenty-five years. First of all, the progress of physics has involved a conflict—that seemed first to be inextricable—between the classical mechanics of Galilei, Huygens and Newton, and the electromagnetic theories of Maxwell, Hertz and Lorentz, and so has brought into question the fundamental principles of natural philosophy. At the same time, the discovery of new elements having paradoxical properties, the study of new radiations, of the Brownian movement, rekindled all the controversies relating to the atomic and energetic theories and obliged the scientists to make a new survey of the principles of chemistry and to revise all their ideas about the constitution of matter. Lastly, the experiments of the biologists and the

exhumation of Mendel's ideas brought about a crisis of the transformist theories and made it necessary to reexamine all our ideas concerning the evolution of life.

However, if the philosophic revival which is now going on has been principally caused by the progress of science and only began in this century, the movement that slowly prepared it is obviously older and more complex. One must first take into account all the scientific work of the last century. This was perhaps less revolutionary and did not provoke sharp crises, like the discoveries just alluded to, but none the less it obliged scientists to modify and to elevate their point of view. Besides, it must be remembered that the writings of some of the scientists of the nineteenth century, namely Helmholtz, Claude Bernard, Berthelot, were already of a synthetical type. But a philosophic school has also in a great measure contributed to this renaissance: I refer to the positivist school represented in France by Auguste Comte, and in England by Herbert Spencer. Our own endeavors are certainly a direct result of their activity. One might say that the positivist ideas have never been better understood nor more popular than they are now. But we must not be led astray by this. It is only since the progress of science has extenuated at the same time the dogmatism and the agnosticism of the first positivist school, and made its ideals broader and more flexible, that positivism bears all its fruit.

This is the first evolution the explanation of which was necessary to show the origin of our ideas. Resounding discoveries determined very grave crises in many departments of science, and so gave a new scope to the philosophic studies that had been despised for a long time. This new philosophy is simply the old positivism, made more supple and more realistic. This is very remarkable, indeed, because the positivist philosophy that had been built up for the very use of scientists had at first not been able to

triumph over their indifference; its success was not secured until the whole structure of knowledge had been shaken and endangered by the very progress of science.

But this is not all. There is still another crisis that seems to have just reached its climax. The triumph of positivism was a triumph rather for science than for philosophy. Many people thought that philosophy would soon be incorporated into science. It would be a philosophy of science, it would gravitate around scientific facts and ideas, or it would not be at all. Its function would be to think out science, nothing more. Such exaggerations, such a misunderstanding of philosophy's historical role,—namely, to be an independent vanguard, a storehouse of general and leading ideas extracted not only from science but from the whole of human experience,—could not help bringing about a new reaction. This reaction is the intuitionism of Bergson, the radical empiricism of William James, the humanism of F. C. S. Schiller, the instrumentalism of John Dewey. I shall simply call it the pragmatist movement. By loudly asserting the claims of intuition, it asserted at the same time the rights to existence of a philosophy independent of the positive sciences. That is the only point of concern to us. And it is so much the more necessary to lay stress upon it, that, in my opinion, it is the best way to show that the conflict between neo-positivists and pragmatists, if it is partly irreducible, is, notwithstanding that, much less grave than it might appear at first sight. For one thing, we must bear in mind that we have all—philosophers, historians, scientists—the same purpose: we try to explain, to generalize, to deepen, to simplify the data of experience. And our very methods have very close analogies: all our knowledge is to a certain extent scientific knowledge, and the pragmatist himself assumes a scientific attitude when he scrutinizes his intuitions. Moreover, would the deep cause of the conflict between the positivist

and the pragmatist points of view not lie in the very complexity of our intellectual needs? These needs are of a practical, utilitarian nature and at the same time of a theoretical, esthetic nature; we need to think and to understand, but, at the same time, we need to act. Would it not lie also in the complexity of the problems raised by ever changing life? Indeed, does not life sometimes oblige the most determined agnostics to reason like pragmatists, and reciprocally? It is owing to these deep causes, inherent in our own nature and in the nature of things, that these antagonistic points of view evidence themselves and clash during the whole development of human thought. It may be well, indeed, to remember that if the pragmatist theories have appeared in a new and fascinating shape, thanks to the genius of Bergson and James, they are as old as science itself.

It is necessary to make these remarks to show that we have not to trouble ourselves too much about this crisis. Besides, positivists and pragmatists all agree in respecting science and all acknowledge the necessity of knowing it as well as possible and of having continual recourse to it. It is of the utmost concern to all of them to study the principles and the history of science. Therefore, we do not care much for their quarrels; we simply accept and record them as interesting human facts, as a new evidence of our mind's complexity.

In short, scientists and philosophers are at the present time unanimous in wishing that the general tendencies and fundamental principles of science be constantly extricated, criticized and stated with more precision. They are well aware that it is now an essential condition of progress and security. But how will it be possible to conciliate the imperious needs of synthesis and the division of labor?

It would seem that the only possible solution is that which was recommended by Auguste Comte and partly

realized by himself and his disciples: namely, to originate a new great specialty, the study of scientific generalities. To secure the unity of knowledge it will be more and more necessary that some men make a deep study of the principles and of the historical and logical development of all the sciences. Of course, they will not be expected to be perfectly acquainted with all the technical details, but they must have at their command a thorough knowledge of the great lines and of the cardinal facts of each science. It is a very difficult but not an impossible task. The inconveniences of excessive specialization will be happily counterpoised by this new branch of knowledge, which induces a collaboration of philosopher, historian and scientist. It will clearly appear from the following pages that the best instrument of synthesis, and the most natural hyphen between scientist and philosopher is the history of science.

THE HISTORY OF SCIENCE.

Auguste Comte must be considered as the founder of the history of science, or at least as the first who had a clear and precise, if not a complete, apprehension of it. In his *Cours de philosophie positive*, published from 1830 to 1842, he has very clearly brought forward the three fundamental ideas which follow: (1) A synthetic work like his cannot be accomplished without having constant recourse to the history of science; (2) It is necessary to study the evolution of the different sciences to understand the development of the human mind and the history of mankind; (3) It is insufficient to study the history of one or of many particular sciences; one has to study the history of all sciences, taken together. Besides this, as early as 1832, Auguste Comte made an application to the minister Guizot for the creation of a chair, devoted to the general history of sciences (*histoire générale des sciences*). It was sixty years before this wish of his was granted, and the course

entrusted to Pierre Laffitte was inaugurated at the Collège de France in 1892, thirty-five years after Comte's death. Another French philosopher, Antoine Cournot, also contributed to the clearing up of our ideas, namely by the publication in 1861 of his book *Traité de l'enchaînement des idées fondamentales dans les sciences et dans l'histoire*. However the real heir to Comte's thought, from our special point of view, is neither Laffitte nor Cournot, but Paul Tannery. It is hardly necessary to say much of him, because all who have the slightest knowledge of the history of science must needs have come across one of his numerous memoirs, all so remarkable for their originality and exactitude. Paul Tannery himself attached importance to his intellectual connection with Comte and often expressed his admiration for the founder of positivism.

Tannery's philosophy is very different from Comte's, but the greatest difference between them is that Comte's knowledge of the history of science was very superficial, whereas Paul Tannery, being extremely learned and having at his disposal a mass of historical research work which did not exist in the thirties, knew more of the history of science than anybody else in the world. Certainly no man ever was better prepared to write a complete history of science, at least of European science, than Paul Tannery. It was his dream to carry out this great work, but unfortunately he died in 1904.

One can understand the history of science in different ways, but these different conceptions do not contradict each other; they are simply more or less comprehensive. My own conception does not differ much from Tannery's, except that I attach more importance to the psycho-sociological point of view.

Auguste Comte had noticed all the bonds that unite the different sciences, but he did not give enough attention to them. If he had understood that these interactions and this

interdependence have existed in all directions from the very beginnings of science, would not the rigid framework of his *Cours de philosophie* have been burst asunder?

On the other hand, some people seem to think that it is impossible to write the history of science as a whole, that the subject is too great. I should rather say that the very impossibility is to pick out from this inextricable network the development of one single branch of human knowledge. Moreover it is actually impossible to write the history of any important discovery without writing, willingly or not, a chapter of the history of science—I mean the history of science as a whole. How could we explain, for instance, the discovery of the circulation of the blood if we did not explain the evolution of anatomy, of comparative zoology, of general biology, of natural philosophy, of chemistry, of mechanics? Likewise, to make clear how they succeeded by degrees in determining longitudes at sea, one has to resort to the history of pure and applied mathematics, the history of astronomy and navigation, the history of watch-making, etc. It would be easy enough to give more examples of the same kind.

Further, it is only by considering the history of science as a whole that one can appraise the scientific level of a definite period or of a definite country. It has happened more than once indeed that one science became neglected while others were thriving, or that scientific culture moved from one country to another. But the historian is not deluded by these facts, and he does not think that human genius is suddenly quenched or rekindled; from his synthetical standpoint he sees the torch of light pass from one science to the other or from one people to another. He perceives better than anybody else the continuity of science in space and time, and he is better able to estimate the progress of mankind.

But the historian's mind is not satisfied with the study

of the interactions between the different sciences. He wishes to study also the interactions between the different sciences on one hand and all the other intellectual or economic phenomena on the other hand. As a matter of fact he has to give a great deal of attention to these reciprocal influences, but of course he does not forget that the aim of his work is essentially to establish the interconnection of scientific ideas.

In short, the purpose of the history of science, as I understand it, is to establish the genesis and the development of scientific facts and ideas, taking into account all intellectual exchanges and all influences brought into play by the very progress of civilization. It is indeed a history of human civilization, considered from its highest point of view. The center of interest is the evolution of science, but general history remains always in the background.

It follows from this definition that the only rational way to subdivide this history is not at all to cut it up according to countries or to sciences, but only according to time. For each period of time, we have to consider at once the whole of its scientific and intellectual development.

Of course to make this general synthesis possible, it will often be expedient, or even necessary, to write monographs or partial syntheses of different kinds. For instance, the study of the archives of a definite place leads naturally to the drawing up of an essay on the history of science in that place. On the other hand, a specialized scientist will be tempted to look up the genealogy of an idea in which he is particularly interested, or to write the biography of a forerunner whose work and genius he can better appreciate than anybody else. But all this research is necessarily incomplete and does not acquire its proper significance so long as it cannot be properly inserted into a history of the sciences of the same age. It may be worth while to add that all monographs are not equally useful; some are so

clumsy and absurd that they obscure, mislead and delay the next synthesis.

To elaborate our historical work we have, of course, to use the same methods that are used by ordinary historians to appraise and criticise the materials available to them. But the historian of science has to use, independently, some other methods of a more special nature. I cannot explain them here, but it is easy to understand that, for instance, to establish at what date a discovery became a real part of science and enriched human experience, the historial exegesis must be supplemented by a scientific exegesis, based on the evidence given by the positive sciences.

We must try to marshal all scientific facts and ideas in a definite order; this means that we must try to assign to each of them a date as precise as possible—not the date of their birth or of their publication, but that of their actual incorporation into our knowledge. Likewise biographers have to exert themselves to fix precisely during which periods the influence of great scientists was the most felt, in order to range them in chronological series. That is generally a very difficult thing to do, and the reader will not fail to appreciate the work that is discreetly accomplished by such scholars. This work of erudition is the bed-rock on which all historical writing is built up.

These remarks complete and add precision to our definition of the history of science. However it may be well to give some more details about the different exchanges which the historian has to consider to put the evolution of science in its proper light.

I shall successively examine some of the other departments of life which are the most interesting for the historian of science: (1) General history or the history of civilization; (2) The history of technology; (3) The his-

tory of religions; and (4) The history of fine arts and arts and crafts.

1. *Science and Civilization.* Since the eighteenth century, and notably under the influence of Vico, Montesquieu and Voltaire, the conception of history has become more and more synthetical. History, the principal interest of which consisted in military records and court annals, is growing up into a history of civilization. It stands to reason that a sufficient knowledge of the history of civilization is absolutely necessary, were it only to locate the scientific facts in the very surroundings that gave rise to them.

On the other hand the historian of civilization can no longer remain unacquainted with the history of science. Some of the most recent historical manuals contain paragraphs devoted to it. It is true, the space allowed is rather scanty, but that is a beginning. I feel confident that before long general histories will be written where the history of science, far from being banished to some obscure corner, will be, on the contrary, the very center of the picture. Is not science the most powerful factor of evolution?

Some examples will illustrate the signification of the history of civilization: How can one account for the fact that the Latin manuscripts containing the translations of Greek authors made from Arabic texts, have so long barred the way to the printed translations that had been elaborated direct from the Greek texts? The latter, indeed, were much better. Björnbo has given some reasons that are very probably the true ones. The printed books that nobody cared to copy, became rarer and rarer. On the other hand the manuscripts were copied over and over again and continually multiplied. Besides, the copyists lacked knowledge and critical sense to a great extent, and they could not help being favorably impressed by the bulk of

Arabic literature. Mere scientific reasons do not suffice to explain the creation of the metrical system by the French revolutionaries. This creation was also in part a reaction against the "foot of the king" of the *ancien régime*. Financial or tariff regulations or the promulgation of labor laws can transform the business life of a country and, indirectly, its scientific production. To understand the beginnings and development of geography one has to take into account many facts that are quite foreign to science. For instance: the quest of mythical treasures; conquerors' ambitions; religious proselytism; the adventurous instincts of daring young men. Lastly, it is necessary to know the history of epidemics and to study all the social facts that have been their causes or their results, to correctly estimate the evolution of medical ideas.

2. *Science and Technology.* Industrial requirements are always putting new questions to science, and in this way they guide, so to say, its evolution. On the other hand the progress of science continually gives birth to new industries or brings new life into old ones. It follows that the history of science is constantly interwoven with the history of technology, and that it is impossible to separate one from the other.

Let us see some examples. After exhausting-pumps had been invented there was such a demand for good pumps of this kind that special workshops were founded in the beginning of the eighteenth century, in Leyden, Holland, to make them, and of course these workshops soon undertook to make other scientific instruments. It is hardly necessary to point out how much the making of these instruments is intimately connected with the history of physics or astronomy.

A geological discovery suffices to revolutionize a whole country and transform an agricultural nation into an in-

dustrial one. Of course a transformation as complete as this involves a radical change in scientific needs. The working of mines has always exerted such a deep influence on the evolution of science and civilization that one might compare the importance of mines in the history of science with that of temples in the history of art. L. de Launay has very clearly shown that the silver mines in Laurion played a considerable part in the history of Greece.

The history of chemistry would sometimes be unintelligible if the history of chemical industries was not studied at the same time. Let me simply remind the reader of the case of coloring matters. Industrial research made in this direction has deeply influenced the progress of organic chemistry. On the other hand it is well known how much has been done to improve this industry by the scientists of the German Chemical Society.

A chemical discovery can revolutionize a whole country, just as completely as a geological one; as soon as it becomes possible to realize, on a business basis, the chemical synthesis of a natural product (like indigo, vanilla, India rubber), the agricultural industry and civilization of immense countries will be in danger.

Technical inventions are every day more precisely determined by industrial needs. The manufacturer can often say very definitely to the inventor: "This is the invention which I now need to improve my production." Besides, every invention starts a series of others that the first has made necessary and that it would have been impossible to realize, or even to conceive, before.

Lastly, commercial needs also influence the development of the sciences, not only the natural sciences and geography (that is too obvious to dwell upon), but even mathematics. It is necessary to take into account the evolution of bookkeeping and banking business to thoroughly understand the introduction and the spread of Hindu-

Arabic numerals into Europe, and later the invention of decimal fractions. It is also a great deal owing to commercial requirements that many astronomical discoveries were made, and that the different systems of weights and measures were created.

3. *Science and Religion.* Science and religion never ceased to influence one another, even in our own time and in the countries where science has reached a high degree of perfection and independence. But of course the younger science was, and the farther we go back through the ages, the more numerous these interactions are. Primitive people cannot part scientific or technical ideas from religious ones, or, more exactly, this classification has no sense to them. Later, when the division of labor had created some scientists or engineers, distinct from the priests, or at least had given birth to a class of priests who had undergone a higher scientific training than their colleagues, even then the interpretation of the holy books, the observance of rites, the needs of agriculture and medicine, the making of the calendar, and above all, the hopes, the fears and the anxieties of a very precarious existence, have been innumerable links between science and religion. The great plagues, and generally all cataclysms, for instance earthquakes or wars, have been followed by religious revivals and often by violent outbursts of religious fanaticism.

On the other hand I know many cases where the priests themselves have been the transmitters of knowledge from one generation to the following. The best example of this can be found during the period extending from the end of the second school of Alexandria to the ninth century. We owe, if not the advancement of science, at least its conservation, to the Fathers of the Latin church and to the Nestorian heresy.

In some other cases the influence of religion is less

direct, but not less important. For instance A. de Candolle has proved that the Protestant families which were exiled from the Catholic countries of Europe during the sixteenth and seventeenth centuries and even during the eighteenth, have given birth to an extraordinarily high number of distinguished scientists. That is not to be wondered at. These people who preferred the misery of exile to moral servitude, were certainly above the average as to their conscientiousness and earnestness.

The interactions between science and religion have often had an aggressive character. There has been most of the time a real warfare. But as a matter of fact it is not a warfare between science and religion—there can be no warfare between them—but between science and theology. It is true that the man in the street does not easily differentiate between religious feelings and faith, on one side, and dogmas, rites and religious formalism on the other. It is true also that the theologians, while affecting that religion itself was aimed at when they alone were criticized, have not ceased from aggravating these misunderstandings. An excellent proof of this has been given in this country. One of the great men of these United States, Andrew Dickson White, has published a splendid book on *The Warfare Between Science and Theology*. Mr. White is a very godly man, and his book is; it is hardly necessary to state, extremely liberal and indulgent to everybody. Notwithstanding this, the author and his book had to bear the attacks of a great many fanatics.

One of the saddest results of these misunderstandings is that some very religious and sincere souls have been misled and have treated science as an enemy. Another important result is that the evolution of science is very intimately interwoven with that of religions and their heresies.

4. *Science and Art.* It may be useful to tender some

remarks upon the different characteristics of scientific and artistic work before pointing out what is interesting from our point of view in the history of art. In the history of art as it is generally taught, very little is said about technicalities. Are there many people who know, or care to know, what kind of colors Botticelli used, or what were the tools of Phidias? We love a work of art for itself. It is essentially the ultimate result that interests us, not the methods employed to obtain it. On the contrary in the domain of learning the result is generally less interesting than the methods employed to reach it.

The history of science is not merely a history of the conquests of the human mind, but it is much more a study of the instruments—material and intellectual instruments—created by our intelligence; it is also a history of human experience. This long experience of the past has much more significance for the scientist than for the artist. The artist admires the work of his forerunners, but the scientist does more than admire, he makes actual use of it. The artist may find an inspiration in it, but the scientist tries to incorporate it entirely in his own work. It is very difficult to conceive progress in art. Does Rodin carve better than Verrochio or Polycletus? The pictures by Carrière, by Watts, or by Segantini, are they finer than those by Fra Angelico, by Van Eyck or by Moro? Have these questions even any sense?

In the domain of science the matter is quite different. Undoubtedly it would be foolish to discuss whether Archimedes was more or less intelligent than Newton, or Gauss; but we can in all security assert that Gauss knew more than Newton, and that Newton knew more than Archimedes. The making of knowledge, unlike that of beauty, is essentially a cumulative process. By the way, this is the reason why the history of science should be the leading thread in the history of civilization. Nothing that has been done or

invented gets lost. Every contribution, great or small, is appreciated and classified. This cumulative process is so obvious that even very young men may be better informed and more learned than their most illustrious forerunners. As a matter of fact they are standing on the shoulders of their predecessors, and so they have a chance to see farther. If they are not very intelligent they may be inclined to think that it is useless to study history, under the misapprehension that they already know of the past all that is really worth knowing. In short, we are not sure that men become more intelligent—that is, whether intelligence increases—but we positively know that human experience and knowledge grow every day. As I have said, one does not pay much heed to mediocre artists. What they do has not much importance. On the contrary, in the laboratories, libraries and museums where science is slowly growing,—like an ever-living tree,—enormous quantities of excellent work is done by thousands of men who are not unusually intelligent, but who have been well trained, have good methods and plenty of patience.

Scientific work is the result of an international collaboration, the organization of which is perfected every day. Thousands of scientists devote their whole lives to this collective work—like bees in a hive—but their hive is the world. This collaboration does not take place simply in space, but also in time; the oldest astronomical observations are still of some use. Perhaps this collective nature of scientific work is one of the causes of the general indifference concerning its history—indifference strongly contrasting with the widespread curiosity about the history of literature and the fine arts. Science aims at objectivity; the scientist exerts himself to decrease to a minimum his “personal equation.” Works of art on the contrary are extremely individual and passionate, so it is not to be wondered at that they excite more sympathy and interest.

The history of the fine arts and of literature is generally considered as a history of the great artists and of the works they have bequeathed to us. But one could adopt a different point of view: just as the history of science gives us the materials of an evolution of human intellect, so one could look in the history of the arts and of literature for the story of the evolution of human sensibility. The history of science is a history of ideas; just so the history of art could be considered as a history of man's dreams. Understood in that way, the two histories complete and enlighten one another.

The interactions between science and art have been particularly vivid at the times of naturalistic reactions against scholastic and pedantic excesses. It would be extremely interesting to make a closer study of the rhythm of the different tendencies that swayed plastic arts and music, and to look for similar rhythms in the contemporary succession of scientific theories, or more exactly, attitudes. The interference of some men of genius, who were at one and the same time artists and scientists,—such as Leonardo da Vinci, Albrecht Dürer and Bernard Palissy,—gives us a splendid opportunity to study these interactions in their deepest and most fascinating form. On the other hand it is a fact that scientific ideas have often been transmitted by works of art; moreover for all the period that precedes the beginnings of popular printing these works of art give us direct testimonies—often the only ones we have—of inestimable value. For instance it would be impossible to trace the history of ancient chemistry but for all the works of art and decoration that have come to us; and, to understand the history of chemistry, not only in ancient times but even as far as the threshold of the seventeenth century, it is still necessary to study the development of the arts and crafts,—the art of the potter, of the glassmaker, of the

chaser, of the jeweler, of the miniature painter, of the enameler.

But the history of art helps us, above all, to understand the spirit and the soul of vanished civilizations. From this point of view, works of art have an immense superiority over every other manifestation of the human mind; they give us a complete and synthetical view of times gone by; they offer us the information that we need at a glance; they bring the past to life again. A granite sphinx, a Nike, a picture by Giotto or by Breughel, a Gothic cathedral, a mass by Palestrina—all these things teach us more in one flash that living men could do by long discourses.

The following examples will show what kind of information the history of art can give us. It is by comparing Gothic monuments that Viollet le Duc has been able to find out some of the principal commercial roads of the twelfth century. Illustrations from Roman monuments give us exact information as to the origin of domestic and medical plants. Indeed it is through Greece and Rome that most of them were introduced from the East into Europe. The history of these plants tells us all the vicissitudes that modified the commercial and intellectual relations between those peoples. Here is another very curious fact. The great botanist H. de Vries has discovered the variety *monophylla* of *fragaria vesca* in a picture by Holbein the Elder ("The Saint Sebastian of Munich," dated 1516). This variety is now cultivated in botanic gardens as a rarity. One guesses that similar discoveries, however small they may appear, sometimes accomplish the solution of historical problems.

Lastly, I wish to note that the history of science is also, to a certain extent—perhaps less than some mathematicians think, but much more than the artists suppose—a history of taste. Leaving aside the external beauty of many books of science, for many scientists were splendid

writers (think of Galilei, Descartes, Pascal, Goethe, Darwin), the very substance of their work has often a great esthetical value. Scientists, who are men of taste, very easily distinguish the scientific theories that are beautiful and elegant from the others. It would be wrong to ignore this distinction, because this beauty and harmony, that common people cannot see but that the scientist does see, is extremely deep and significant. One might ask: "These theories that are more beautiful—are they more true?" Anyhow they are easier and more fertile; and for that reason alone it is worth while to give them our preference.

THE SCIENTIFIC POINT OF VIEW.

The history of science has a great heuristic value, especially if it has been worked out by somebody who is well acquainted with modern scientific tendencies as with ancient ones. The sequence of old discoveries suggests similar concatenations to the scientist, and so enables him to make new discoveries. Disused methods, cleverly modified, may be rendered efficient again. When it is understood in this way, the history of science becomes really a research method. A great scientist of our own time, Ostwald, has even gone so far as to say that "It is nothing but a research method." We do not admit this much. Anyhow, new and old science complete and continuously help one another to advance and to diminish the unknown that surrounds us everywhere. Does this idea not illuminate our conception of the universal scientific collaboration? Death itself does not interrupt the scientist's work. Theories once unfolded are eternally living and acting.

To give to our history all its heuristic value, it is not sufficient to retrace the progress of the human mind. It is also necessary to remember the regressions, the sudden halts, the mishaps of all kinds that have interrupted its

course. The history of errors is extremely useful; for one thing, because it helps us to better appreciate the evolution of truth; also because it enables us to avoid the same mistakes in the future; lastly, because the errors of science are of a relative nature. The truths of today will perhaps be considered tomorrow, if not as complete mistakes, at least as very incomplete truths; and who knows whether the errors of yesterday will not be the approximate truths of to-morrow? Similar rehabilitations frequently occur, and the results of historical research often oblige us to admire and honor people who have been misunderstood and despised in their own time. This incidentally proves to us that the study of the history of science has also some moral advantages.

However the history of superstitions and errors must not make us forget that it is the history of truth—the most complete and the highest truths—that interests us primarily. Besides, one may aim at retracing the history of truth in its entirety, because it is naturally limited, but the history of errors is infinite. It is thus necessary to fix some artificial limits to the latter and to choose judiciously between the errors and superstitions. A great simplification is obtained by classifying the errors in groups. Indeed the same mistakes and superstitions appear over and over again in different shapes, and it is useful to know the different types of errors to understand the mechanism of intellect.

It is much to be regretted that many scientists decline to admit the utility of historical research or consider this simply as a kind of pastime of small importance. They base their contempt on the following argument: "All the best of ancient science has been assimilated and incorporated in our own science. The rest did not deserve more than oblivion, and it is awkward to overburden our memory with it. The science that we are learning and teaching

is the result of a continuous selection which has eliminated all the parasitic parts in order to retain only that which is of real value."

It is easy to see that this argument is not sound. For one thing, who will guarantee that the successive selections have been well made? This is so much the more a matter of doubt that this selective and synthetic work is generally done not by men of genius, but by professors, by authors of textbooks, vulgarizers of all kinds, whose judgment is not necessarily irreproachable and whose intuitions are not always successful. Besides, as science is constantly evolving, as new points of view are introduced every day, any idea that has been neglected may be considered later on as very important and fertile. It often happens also that some facts that were scarcely known all at once become very interesting, because they can be inserted into a new theory that they help to illustrate. Of course scientific syntheses—like those represented by our textbooks—are indispensable. Without them science could hardly be transmitted from one generation of scholars to the next, but it must be understood that they are always provisional and precarious. They must be periodically revised. Now how would that be possible if the history of science did not show us our way through all the unutilized experience of the past? History is, so to say, the guide—the catalog—without which new syntheses and selections made from fresh points of view would hardly be possible. All the vicissitudes and recantations of science prove conclusively that no man can ever flatter himself that he has definitely and completely exhausted a scientific fact or theory. No particle of human experience, however small, can be entirely neglected. To assert this is to assert, at the same time, the necessity of historical research.

Moreover among scientific works there are some, the genesis of which cannot be explained in the ordinary ana-

lytical way. They introduce abrupt discontinuities into the evolution of science because they so far anticipate their own time. These works of genius are never entirely known, and the interest they offer is never entirely exhausted. It is perhaps because it is almost inexhaustible, that true genius is so mysterious. Sometimes centuries pass before the doctrines of a man of genius are appraised at their true value. A great deal of benefit is still to be reaped from the reading of the works of Aristotle, Diophantus, Huygens or Newton. They are full of hidden treasures. For it is a gross mistake to think that there is nothing more in such works than the facts and ideas which are positively formulated; if that were true it would of course be useless to refer to them, the enunciation of these facts and ideas would suffice. But that is not true, and I cannot but advise those who have any doubt about it, to try. They will find that nothing excites the mind more than this return to the sources. Here also it is the historian's business to point out to the scientist the very sources where he will the most likely invigorate his mind and get a fresh impulse.

I wish now to give a few examples to illustrate the preceding remarks. Any amount of them can be found in the history of medicine; we need but recall how greatly the whole of medical evolution has been influenced by the Hippocratic teaching, our modern ideas on humorism and naturalism; or, again, the organo-therapeutic theories. Not only are the old ideas restored to vogue, but it sometimes seems that a kind of rhythm brings them back to light periodically. Likewise Georges Bohn has shown the periodical return, in the domain of comparative psychology, on one hand, of the animistic and anthropomorphic conceptions, on the other hand, of the positivist conceptions. As a rule the further science is removed from the mathematical form the more likely these vicissitudes. One can also say that when science is more accurate, that is to say, when the domain

of uncertainty and hypothesis becomes narrower, the oscillations of the mind between divergent points of view are so much less numerous,—but they do not cease entirely. Thus E. Belot has recently reintroduced into cosmology, in a very seductive shape, the vortex theory that one would have thought had been entirely banished by Newton's criticisms. Similarly Walter Ritz has given weighty reasons for reinstating into optics the emission theory, which seemed to have been forever exploded by the discoveries of Huygens, Young and Fresnel.

But the best examples of such return to ancient knowledge are given to us by the history of technology. The history of chemical industries is very significant from this point of view. This is due to the fact that economic conditions here play a considerable part. In order that an invention may be realized it does not suffice that it be theoretically possible; it must pay. Now thousands of circumstances continually modify the material factors which the engineer is struggling with; many are of such a nature that nobody could foresee them, or (what amounts to the same thing), that it would cost too much to insure oneself against all of them. If new products appear on the market, or if the prices of some of the raw materials happen to vary considerably, or if new discoveries are made, or if new residues are to be employed, old methods that were too expensive may become economical, or reciprocally. Hence the chemist and the engineer have a vital interest in knowing the processes that have fallen into disuse, but to which the very progress of science may give from one day to the next a new career. The history of science is to them, so to say, what forsaken mines are to the prospector.

But in my opinion, however important its heuristical value may be, there are still deeper reasons why the scientist should give his attention to the history of science. I am thinking of those which have been so splendidly illus-

trated by Ernst Mach in his *Mechanics*. For one thing, it is obvious that "they that know the entire course of the development of science will, as a matter of course, judge more freely and more correctly of the significance of any present scientific movement than they who, limited in their views to the age in which their own lives have been spent, contemplate merely the momentary trend that the course of intellectual events takes at the present moment."² In other words, to understand and to appraise at its just value what one possesses, it is well to know what the people possessed who came before us; this is as true in the domain of science as it is in daily life. It is his historical knowledge that discloses to the scientist his precise attitude toward the problems with which he has to grapple, and that enables him to dominate them.

Moreover while research workers exert themselves to extend the boundaries of science, other scientists are more anxious to ascertain whether the scaffolding is really solid and whether their more and more daring and complex edifices do not risk giving way. Now the task of the latter, which is neither less important nor less lofty than that of discovery, necessarily implies a return to the past. *This critical work is essentially of an historical nature.*³ While it helps to make the whole fabric of science more coherent and more rigorous, at the same time it brings to light all the accidental and conventional parts of it, and so it opens to the discoverer's mind new horizons. If that work were not done, science would soon degenerate into a system of prejudices; its principles would become metaphysical axioms, dogmas, a new kind of revelation.

That is what some scientists come to, who, for fear of falling into literature or metaphysics (as they put it),

² Ernst Mach, *The Science of Mechanics*, translated by Thomas J. McCormack, 2d rev. ed., p. 7. Chicago, Open Court Publishing Co., 1902.

³ See George Sarton, "Les tendances actuelles de l'histoire des mathématiques," *Isis*, Vol. I, pp. 577-589, especially pp. 587-8.

banish all historical or philosophic considerations. Alas! the exclusive worship of positive facts makes them sink into the worst kind of metaphysics—scientific idolatry.

Fortunately it happens at certain periods of evolution that resounding and paradoxical discoveries make an inventory and a thorough survey of our knowledge more obviously necessary to everybody. We are fortunate enough to be living at one of these critical and most interesting periods.

The purpose of historical criticism is not merely to render science more accurate, but also to bring order and clearness into it, to simplify it. Indeed it is the survey of the past that enables us the best to extricate what is really essential. The importance of a concept appears in a much better light when one has taken the trouble to consider all the difficulties that were surmounted to reach it, all the errors with which it was entangled, in short all the life that has given birth to it. One could say that the riches and fertility of a concept is a function of its heredity, and that alone makes it worth while to study its genealogy.

The history of science is accomplishing an endless purification of scientific facts and ideas. It enables us to deepen them, which is undoubtedly the best way to make them simpler. This simplification is of course the more necessary as science grows bigger and more intricate. By the way, it is thanks to this progressive simplification that an encyclopedic knowledge does not become utterly impossible; in certain cases it becomes even more accessible. For instance is it not easier to study chemistry or astronomy—I mean the essentials of it—now than it was, say, in the fifteenth century?

I think one can infer from all the preceding remarks that no scientist is entitled to claim a profound and complete knowledge of his branch if he is not acquainted with its history. I have compared the scientific achievements

of mankind with the collective work that the bees accomplish in their hives. This comparison is particularly apposite to the scientists who have specialized to excess and diligently work in their own narrow field, ignoring the rest of the world. Such men are doubtless necessary, as are the bees that provide honey. But their endeavors could never give birth to a systematic knowledge, to a science proper. It is the more necessary that other scientists raise themselves above the artificial partitions of the different specialties. Their investigations irresistibly lead them to the study of history, and they obtain from it a deeper apprehension of their own collaboration in the grand undertakings of mankind. Just as one experiences gratification by knowing where one is and why, just the same it gives them pleasure to locate their own task in the world's work and to better grasp its relative import. And also, they understand better than the others do the significance of the thousand and one ties that connect them to their fellowmen—and the power of human solidarity, without which there would be no science.

THE PEDAGOGIC POINT OF VIEW.

In many countries one cannot become a teacher at least in the secondary schools, if one has not studied the history of pedagogics. But is it less important to know the history of what is taught? And will not any one who knows this history be better prepared to distinguish what is essential and really interesting from what is not, and to teach his pupils the best of each science? Moreover will the history of science not enlighten the history of pedagogics?

Science is generally taught in a much too synthetic way.⁴ It may be that this method is indeed the best for the average student who passively accepts the master's authority. But those whose philosophical mind is more awake

⁴ My experience refers especially to the European continent and to the teaching of the physical and mathematical sciences.

can hardly be satisfied by this food, the preparation of which is unknown to them. Instead of being assuaged by harmonious order and perfect science, they are devoured by doubt and anxiety: "Why does the master teach us so? Why has he chosen these definitions? Why?" Not that they are loath to use synthetic methods; on the contrary, these young men will probably be the first to admire the depth and elegance of such teaching once they have grasped from their own experience its logical appositeness, its generality and its economy. But first of all they want to know "how all that was built up," and their mind instinctively recoils from a dogmatism that is still arbitrary to them.

It remains arbitrary indeed so long as the reasons that justify and render natural one arrangement in preference to any other, have not been explained. I know that it is not easy to teach beginners in this way, but at least the deficiencies of the present methods could be tempered, and I do not ask for more.

Nothing would be more useful from this point of view than to work out some text-books in which science would be expounded in chronological order; this is indeed a very important task for which Ernst Mach has given us some admirable models. These text-books would not be employed for elementary study, unless the pupils used them at the same time as others composed along dogmatic lines. Students should have to study the latter and read the first. But in my opinion, these historical text-books would especially stand professors in good stead, by enabling them to illustrate their lessons and make them more intuitive. Oral teaching, more pliable than written teaching, would easily admit of short historical digressions. Would the students not more easily remember the abstract truths that are impressed upon them in ever increasing quantities, if their memory could lay hold of some live facts?

But that does not exhaust the pedagogic importance of

the history of science. Nothing is better fitted to awaken a pupil's critical sense and to test his vocation than to retrace to him in detail the complete history of a discovery, to show him the trammels of all kinds that constantly arise in the inventor's path, to show him also how one surmounts them or evades them, and lastly how one draws closer and closer to the goal without ever reaching it. Besides, this historical initiation would cure the young students of this unfortunate habit of thinking that science began with them.

Good scientific biographies have also a great educational value; they lead an adolescent's imagination in the best direction. Critical and sincere biographies make excellent contributions to the history of mankind. And would not the students work with a better heart and more enthusiasm, would they not have a deeper respect for science, if they knew a little more about the heroes who have built it up, stone by stone, at the expense of so much suffering, struggle and perseverance? Would they not be more eager to undertake some disinterested research work? Or at least would they not better appreciate the greatness and beauty of the whole if they had, more or less, partaken of the joy and intoxication of seeing it accomplished amidst continuous difficulties?

Lastly, the history of science—even more than ordinary history—is a general education in itself. It familiarizes us with the ideas of evolution and continuous transformation of human things; it makes us understand the relative and precarious nature of all our knowledge; it sharpens our judgment; it shows us that, if the accomplishments of mankind as a whole are really grand, the contribution of each of us is in the main small, and that the greatest ought to be modest. It helps to make scientists who are not mere scientists but also men and citizens.

THE PSYCHOLOGIC AND SOCIOLOGIC POINTS OF VIEW.

The history of science, its birth, its evolution, its diffusion, its progress and regressions, irresistibly imposes upon us a series of psychological problems. We here enter the field of universal history, such as the much lamented Karl Lamprecht has defined it; for the history of science in the main amounts to psycho-sociological investigation.

It is necessary here to make a preliminary distinction. The progress of science is due to two different kinds of causes: (1) Purely psychological causes, the intellectual work of the scientist; (2) Material causes, principally the appearance of new subject matter or the use of improved scientific tools. Of course it is not difficult to show that the origin of these material causes is itself more or less of a psychological nature. But the distinction holds good; a discovery has not indeed the same character, the same psychological importance, if it is the almost automatic result of a technical improvement, as if it is the fruit of a mind's reaction. We propose to discover the general laws of the intellectual evolution of mankind, if such laws exist. These studies might also help us to better understand the intellect's mechanism. But of course we have given up the extravagant idea of establishing *a priori* the conditions of scientific development. On the contrary our end is to deduce them from a thorough analysis of all the accumulated experience of the past.

The best instrument for these studies is the comparative method, and this means that we must not expect to reach a degree of accuracy of which this method does not admit. But no scientific work would be possible in the domain of biology and sociology if one did not have the wisdom and patience to be satisfied with the approximation that is within one's reach. The comparisons may be confined to the realm of science; I would call these the internal com-

parisons. They may also be made between the evolution of scientific phenomena and that of other intellectual or economic phenomena; and these I would call the external comparisons. The greatest difficulty of course is to find evolutionary processes that can be adequately compared and that are sufficiently independent one of another.

The application of this method has already supplied some results which have been very improperly called "historical laws," and the exactitude of which is very variable. The following are some examples which I shall refrain from discussing: Paul Tannery has shown that the development of calculus generally precedes that of geometry. In their choice of decorative elements primitive peoples always pass from animals to plants; they never do the contrary. The hypothesis that has been expressed about the course of civilization from the South and the East to the North and the West, is well known. Remember also the law of historical periods proposed by Lamprecht, and especially the famous law of the three states (*la loi des trois états*), formulated by Auguste Comte. The theory of historical materialism, originated by Karl Marx, which has exerted such a deep influence on the thought of the nineteenth century, is also a proper example.

It is sensible to undertake the study of intellectual activities in the same way as we study the industry of the beavers or the bees. Of the work produced by the human brain we generally know nothing but the results, but these are tangible and can be, if not actually measured, at least compared and appraised with more or less precision. The invention of a machine or the discovery of a natural law, are these not at the bottom phenomena of the same kind as the behavior of a crab or of a sea anemone under determined circumstances? They are, of course, much more complex and their study requires the use of new methods, scarcely explored; but can one not admit, at least as a

working hypothesis, that they do not essentially differ? The psychology of the superior functions of the brain is not necessarily more complicated than that of the inferior functions; I should be rather inclined to think the contrary. For instance would it not be easier to retrace the development of a scientific idea in a clear mind than to disentangle, in the prelogical head of a primitive man, the obscure roots of his instinct of property or imitation?

It is from the comparison of these intellectual facts, as they can be collected by the historian of science from the whole intellectual experience of the world, that we may try to deduce the laws of thought. Human experience has been continuously increasing during the ages, but the intellect itself,—has it evolved? The methods of discovery, the mental experiences, the hidden mechanism of intuition—have they not remained somewhat the same? Is there nothing invariable in men's intellectual behavior? What are those invariants, or at least those relative invariants, those more stable parts of our self? To what extent does the scientific environment exert its influence upon the scientists, and *vice versa*? How do social activities evidence themselves in the domain of science? By what mental processes are the ideas of the initiators integrated in the collective thought, to become, by and by, common notions? All these questions, raised by the history of science, are so many psychological problems.

As to research concerning the psychology of invention, choice materials will be found in the history of technology. The results of technical invention are material objects of a very concrete and tangible nature. Besides, the mechanism of industrial discoveries is especially interesting, because to materialize his ideas the engineer has actually to struggle with all the difficulties of real life. The struggle is more obvious here than in any other domain. It happens that unexpected obstacles are so great that the idea cannot

be carried out; but it also happens very often that the very clash of these obstacles gives birth to new ideas, deeper and richer than the original ones. Then one sees, so to say, the invention gush out from the conflict between matter and spirit.

It would be apposite here to present some remarks about the "genealogical" research work that was initiated by Francis Galton and Alphonse de Gandon. These very interesting historico-statistical investigations, intimately connected with the eugenic movement, bring new testimonies to the importance of the history of science from the psycho-sociological point of view. But to give a good idea of these studies I should be obliged to make too long a digression from my subject. I simply refer the reader to my previous publications on these matters.⁵

THE HUMANISTIC POINT OF VIEW.

A deeper knowledge and a greater diffusion of the history of science will help to bring about a new "humanism." (I beg the reader to excuse me for using a word that has already been employed in at least two different senses, but I do not find any other that is more adequate to the idea I wish to convey.) The history of science, if it is understood in a really philosophic way, will broaden our horizon and sympathy; it will raise our intellectual and moral standards; it will deepen our comprehension of men and nature. The humanistic movement of the Renaissance was essentially a synthetic movement. The humanists were longing for a new atmosphere and a broader conception of life; their curiosity was insatiable. We have at least this much in common with them. We know also that if science were abandoned to narrow-minded specialists it would soon degenerate into a new kind of scholasticism, without life

⁵ George Sarton. "L'histoire de la science," *Isis*, Vol. I, pp. 39-41; also, same author, "Comment augmenter le rendement intellectuel de l'humanité?" *Isis*, Vol. I, pp. 219-242, and pp. 416-473 (unfinished).

or beauty—false and wrong like death itself. This would be another good reason for comparing our task with that accomplished by the former humanists. However their movement was essentially toward the past; ours is much more a movement toward the future.

Science, divided into water-tight compartments, makes us feel uneasy;—a world split into selfish and quarrelsome nations (similar to the Italian and Flemish municipalities of the Renaissance) is too narrow for us. We need the full experience of other countries, of other races; we need also the full experience of other ages. We need more air!

It may be useful to lay some stress on the significance of science from the international point of view. Science is the most precious patrimony of mankind. It is immortal. It is inalienable. It cannot but increase. Does not this priceless patrimony deserve to be known thoroughly, not only in its present state but in its whole evolution? Now most men—most scientists—are unfamiliar with the glorious history of our conquests over nature. Would it not be a great work of peace and progress to bring them to better understand and appreciate this intellectual domain which is privileged among all others, *because it is the only one that is entirely common to all?* Science is not only the strongest tie, but it is the only one that is really strong and undisputed.

Science makes for peace more than anything else in the world; it is the cement that holds together the highest and the most comprehensive minds of all countries, of all races, of all creeds. Every nation derives benefit from the discoveries that have been made by the others. There can be no warfare between high-minded scientists.

The further science progresses, the more its international character asserts itself—and this in spite of all jingo-

ist and imperialist tendencies that may occasionally blind and lower some of its servants.

Just as scientific methods are the basis of well-nigh all our knowledge, just so science appears more and more as the bedrock on which every organization has to be built up to be strong and fertile. It is the most powerful factor of human progress. As Mach has perfectly put it: "Science has undertaken to replace wavering and unconscious adaptation by a methodical adaptation, quicker and decidedly conscious." It is the historian's duty to evidence all the scientific facts and ideas that make for peace and civilization; in this way he will better secure science's cultural function.

The international significance of the history of science has not been thus far better grasped for the simple reason that very few historical studies have been inspired by a real international spirit. For one thing universal histories have been almost exclusively devoted to the achievements of the Indo-Aryan race. Everything in them gravitates round the development of Europe. Of course this point of view is absolutely false. The history of mankind is too obviously incomplete if it does not include, on the same level as the Western experience, the immense experience of the East. We badly need the knowledge and wisdom of Asia. They have found other solutions to our own problems (the fundamental problems cannot but be the same), and it is of the greatest importance to consider these solutions, and to consider them in a humble way. It is a fact that they have very often been nearer to truth and beauty than we. Besides, although the development of the Far Eastern countries has been to a great extent independent of our own, there have been far more exchanges, especially in ancient times, than is generally believed, and it is also of paramount importance to investigate these matters.

The progress of mankind is not simply an economic

development; it is much more an intellectual unfolding, as Henry Thomas Buckle has shown with so much force. The whole course of civilization is marked by the triumph of the mental laws over the physical—a triumph of man over nature. To the best of my judgment Buckle has even gone too far in this direction. I am not ready to concede, as he has done, that the changes in every civilized people are dependent solely on three things: (1) The amount of knowledge of the ablest men; (2) The direction of this knowledge; (3) Its diffusion. If Buckle were right all history would be included in the history of science. There are other things to consider.

Moral factors do not deserve to be despised as much as Buckle did, and I think that it is even possible to construct an ethical interpretation of history. To give a moral significance to history the essential condition is to make it as complete, as sincere as possible. Nothing is more demoralizing than histories *ad usum Delphini*. We must display the whole of human experience, the best and worst together. The greatest achievement of mankind is indeed its struggle against evil and ignorance. Nothing is nobler than this endless struggle between the truth of to-day and that of yesterday. It stands to reason that if one side of the picture is not shown—the evil side, for instance—the other loses a great deal of its interest. The quest of truth and beauty is indeed man's loftiness. This is certainly the highest moral interpretation of which history allows.

We must try to humanize science, to better show its various relations with other human activities—its relation to our own nature. It will not lower science; on the contrary, science remains the center of human evolution and its highest goal; to humanize it, is not to make it less important, but more significant, more impressive, more amiable.

The new humanism—as I venture to call the intellec-

tual movement that has been defined in the preceding pages—will also have the following consequences: It will disentangle us from many local and national prejudices, also from many of the common prejudices of our own time. Each age has of course its own prejudices. Just as the only way to get rid of local prejudices is to travel,—similarly, to extricate ourselves from time-narrowness we must wander through the ages. Our age is not necessarily the best or the wisest, and anyhow it is not the last. We have to prepare the next one, and I hope a better one.

If we study history it is not through mere curiosity, to know how things happened in the olden times (if we had no other purpose than this our knowledge would indeed be of a very poor quality); nor is it for the mere intellectual joy of better understanding life. We are not disinterested enough for that. No; we wish to understand, to better foresee; we wish to be able to act with more precision and wisdom. History itself is of no concern to us. The past does not interest us but for the future.

To build up this future, to make it beautiful, as were those glorious times of synthetic knowledge, for instance that of Phidias or of Leonardo da Vinci, it is necessary to prepare a new synthesis. We propose to realize it by bringing about a new and more intimate collaboration between scientist, philosopher and historian. If that were accomplished so much beauty would be given birth to that the collaboration of the artist would also necessarily be secured; an age of synthesis is always an age of art. This synthesis is what I have called “the new humanism.” It is something in the making,—not a dream. We see it growing, but no one can tell how big it will grow.

The writer is convinced that the history of science—that is to say, the history of human thought and civilization in its broadest form—is the indispensable basis of any philosophy.—History is but a method—not an aim.

APPENDIX.

THE TEACHING OF THE HISTORY OF SCIENCE IN THE UNITED STATES.

An elaborate essay on this subject has been published in *Science*, November 26, 1915, pages 746-760, by Frederick E. Brasch ("The Teaching of the History of Science; Its Present Status in Our Universities, Colleges and Technical Schools"). As I shall confine myself to remarks of my own and to only a few extracts from Mr. Brasch's work, the reader who desires to follow up the subject is recommended to read his paper.

To Harvard University belongs the credit of first establishing a course on the history of a particular science: Dr. Theodore W. Richards began as early as 1890, and is still continuing, a course on the history of chemistry. On the other hand the Massachusetts Institute of Technology was the first to recognize the interest of the history of science as a whole: Prof. W. T. Sedgwick and H. W. Tyler have been teaching it in that institution since 1905.

According to Mr. Brasch's painstaking statistics, 162 courses on the history of some special science are now organized in 113 schools. Among them not less than 47 are devoted to the history of mathematics, and not less than 38 to the history of chemistry. Moreover there are 9 courses on the general history of science. To this number could be added 8 temporary courses, namely, Harvard Exchange Lectures, delivered by Dr. L. J. Henderson in five Middle Western colleges, and three courses given by myself at the summer school of the University of Illinois, at the George Washington University in Washington, D. C., and at Clark University.

Mr. Brasch gives the following information about the nine regular courses: (1) Reed College: history forms a part of a course on general science; (2) Lehigh University: "combination of biographies and progress of science"; (3) University of Pennsylvania: the philosophy department has started a historical course entitled "Philosophy of Nature"; (4 and 5) Chicago and Columbia: history of the physical sciences; at the University of Chicago there is a course on the history of science in America; (6 to 9) Harvard, Princeton, the Carnegie and the Massachusetts Institutes of Technology have organized complete courses on the history of the physical and biological sciences.

This information is very meagre. For lectures on a subject that is still so far from being standardized it would be most inter-

esting to know exactly what are in each case the purpose and the methods of the lecturer. It would be interesting also to know how many of these courses have been given by specially trained men and how many have been more or less extemporized by professors already engaged in other fields.

It is worth while to note that Prof. W. T. Sedgwick and H. W. Tyler are preparing a text-book for the use of their own classes. Dr. Walter Libby of the Carnegie Institute of Technology is also preparing the edition of a series of short volumes on the same subject. As the interest in it is now awakening it is likely that many other text-books will appear before long.

I have come to the conclusion that the history of science as a whole, brought at least as far as the eighteenth century and including perhaps some rudiments of this history in our own times, should be taught to all undergraduate students. It would be for them the best scientific introduction, and at the same time it would provide them with a historic and philosophic foundation on which they could build up their special studies. It would open their minds and broaden their horizon from the beginning. Such a course should be taught by some one devoting himself entirely to historical research of this kind. On the other hand the complete history of each science during the last fifty or a hundred years should be studied by all the graduate students, making a special study of the same. This course should be taught by specialists of a quite different kind,—not historians, but scientists, having a sufficient historical knowledge,—generally professors of the school for graduate studies.

It may be objected to my plan that the scientific preparation of most undergraduate students is so scanty that they would not be able to attend these lectures with real profit. In this case it would perhaps be better to reserve them for the graduate students, or to shift them to the very end of the university curriculum. In this second hypothesis the course could be made much more complete and be treated from a much higher point of view. It could be a really inspiring course, giving much food for thought to the best students,—a splendid coronation of their studies. It would open their eyes to the marvelous spectacle of human evolution. It would be for them, before their departure from the university, the great humanistic initiation, the supreme lesson of wisdom, of tolerance and enthusiasm.

Some may doubt whether courses on the history of science are really as necessary as I claim. But one thing is certain: If they are given at all they must be given well. A loose and superficial teaching is worse than none. It would soon bring discredit upon historical studies. We must avoid that at all cost. Therefore it is urgent to organize a seminary in at least one of the universities of this country where normal lessons would be given and the historical methods taught in the experimental way. Those who teach the history of science must needs have a first-hand knowledge of it and be trained to make accurate investigations.

There is no seminary for the history of science in this country, but there is one for the history of mathematics at Teachers College of Columbia University, under the direction of Dr. David Eugene Smith. A splendid library and interesting collections have been formed by him at Teachers College, and original research work on the history of mathematics can be conducted there under the best conditions.

Some seminaries also exist in Europe. I know at least two that are equipped for the study of the history of medicine: the famous Institut für Geschichte der Medizin of Leipsic, so efficiently directed by Dr. Karl Sudhoff, and another one in Vienna under the direction of Dr. Max Neuburger. On the other hand the much lamented A. von Braunmühl founded in Munich a seminary devoted to the history of mathematics; and of course much seminary work was also done in Heidelberg, under Moritz Cantor's direction.

There may be other seminaries which I do not recall; but I know positively that there is none devoted to the history of science as a whole. That is not to be wondered at, as these studies are scarcely begun.

I hope that one of the great American universities will take upon itself this initiative, and organize an institute where all information on the history of science could be centralized, studied and diffused again.

Will America give this great example to the world? I earnestly hope so.

BIBLIOGRAPHY.

The John Crerar Library of Chicago published in January, 1911, "A list of books on the History of Science, prepared by Aksel G. S. Josephson." It is the only list of this kind that I know of, and it is very valuable indeed. However it is far from being complete. For one thing it is simply a list of *books*,

and most historical memoirs are not published in book form. I hear that a supplement is being prepared, and also a companion volume on the History of Industry and Industrial Art. I sincerely hope that the Supplement will contain some critical notes, which allow the reader to make a sensible choice between so many titles. Uncritical bibliographies, where the best and the worst books are all put on the same level, sometimes do more harm than good.

The best way to complete the information given by Aksel G. S. Josephson is to refer to the "Bibliographie critique de toutes les publications relatives à l'histoire, à la philosophie, et à l'organisation de la science," published in *Isis*. Unfortunately this publication has been interrupted by the war, and the last list published (Vol. II, pp. 249-310) was closed in May, 1914. Two other lists were prepared, and one was in the press, when Belgium was invaded. The offices of *Isis* are of course inaccessible. But more copies of the periodical are still obtainable from the publisher for Switzerland and Germany: Max Drechsel, Akademische Buchhandlung, Bern, Switzerland.

It may be useful also to refer to the following article: George Sarton, "Soixante-deux revues et collections consacrées à l'histoire des sciences (Bibliographie synthétique. . . ., I), *Isis*, Vol. II, pp. 132-161 (1915).

GEORGE SARTON.

WASHINGTON, D. C.

THE ANTHROPOLOGY OF THE JEW.

WITH respect to no other people has there been so much hair-splitting controversy as regards classification as with the Jews. Antisemites and philosemites, anthropologists and historians, political reformers and sociologists, Jews and non-Jews, friends and foes alike have all differently defined and described this peculiarly persisting element. Some would call them a race, others a people, still others a religious sect, and so forth. Thus with Chamberlain, Dühring, Wagner, Woodruff and other antisemites the Jews are a race, but distinctly inferior to the so-called Aryan race; with Wirth, Topinard, Weissenberg, Fishberg, Neubauer, etc., they exist only as a social-theological organism; others, as Ripley for instance, would not call them a race but a people, who have only one element in common, and that is a peculiar facial expression.¹ Lazare, on the other hand, would not call them a race, which to him is a misnomer, since no races in the sense of ethnic unities exist, but to him they are a nation, in the sense of unity of sentiments, ideas, and ethics.² Again, Zollschau, Ruppin, Jacobs, Haupt, Andree, Sombart, Salaman, Lucien, Wolf and others believe in the comparative purity of the Jewish race, at least since the time of Ezra, 430 B. C. Zangwill, in a mood of despair, asserts that the Jews exist only as a negative unity, by force of hostile conditions. He says: "No Jewish people or nation now exists,

¹ W. Z. Ripley, *The Races of Europe*, 1899, pp. 368-400.

² B. Lazare, *Antisemitism,—its History and Causes*, 1903, p. 248.

but a multitude of individuals; their only unity being negative; the hostile hereditary vision of the ubiquitous Haman."³

In juxtaposition to this is the difficulty of identifying the Jews with any of the great subdivisions of mankind. The old Semitic affiliation has lately been called into question. Von Luschan, Ripley, Lombroso, etc., are inclined to believe that the Jews are more Aryan than Semitic. Von Luschan emphatically asserts that they are composed of three elements,—the Hittite, the Xantheorous Nordic, whom the present Kurds resemble and who he thinks were affiliated with the Amorites of the Bible, and last the Semitic element; the first two he shows were Aryans.⁴ Haupt, like Von Luschan, believes they have descended from the Amorites, Hittites, and Armenians, but that the Hittites may have been of Mongolian origin. He also informs the writer in a personal letter that he believes that not only the Amorites but the Phenicians also came from Europe.⁵ Judt, cited by Zollschan, on the other hand, thinks the Jews are to be classed with the Alpine races.⁶ Again, there is also the question of the superiority or inferiority of the Jew, which has been so much a point of combat between antisemites and philosemites. Indeed to go into the details of the anthropology of the Jew alone would take us far beyond the scope of this article and would in fact lead us nowhere. We shall content ourselves therefore with establishing a few general facts, and in the light of those facts shall pass the verdict whether or not the Jews are to be considered as a race.

The main fault with the majority of theories lies in their one-sided attitude of partiality. The Jew is not con-

³ I. Zangwill, *The Jewish Race*, 1911, pp. 268-279; G. Spiller (ed.), *Inter-Racial Problems*.

⁴ F. von Luschan, *The Early Inhabitants of Western Asia*, pp. 221-244; *Journ. of Anthropol. Inst. Gr. Br. and Irele.*, N. S., XIV.

⁵ P. Haupt, "Die Juden," *Meyers Konversations-Lexikon*, pp. 328-330.

⁶ I. Zollschan, *Das Rassenproblem*, 1910, pp. 57-58.

sidered collectively as an integral part of an exceedingly complicated organism which we call mankind, but he is measured generally through the horoscope of one's special line of interest. The scientific antisemite, eager to prove his own superiority, considers only that side of the Jew which is below his own standard, underestimating or completely ignoring other phases in which the Jew is markedly above his standard. The same is true of the philosemitic *mutatis mutandis*. So the physical anthropologist considers only the physical side, the economist the economic, the politician the political side, and so forth. Indeed it is only natural to undervalue everything outside of our own line of interest, but none the less faulty. We forget that what makes an individual and a race or people as an aggregate of individuals is an *ensemble* of many things, a totality of physical, psychical, physiological and pathological factors, and it is all of these that have to be considered.

Let us turn now to the above-indicated questions. To begin with the question of the superiority or inferiority of the Jews, we think that the common misconception is partly due to the confusion of the term "inequality" as synonymous with either superiority or inferiority. It is really of inequality of the Jew and non-Jew that we should speak, but inequality does not necessarily mean either superiority or inferiority. We cannot speak of the value of abstract qualities as equal or unequal in the sense of coincidence, as in the case of physical measurements. It is the comparison of the values of those qualities, even though different in kind and nature, that we ought to consider. Two individuals may each excel in one thing; they will be unequal in that their lines of excellence are different, but they are not necessarily superior or inferior to each other, for to society the value of the contributions of each may be of equal importance. In a like manner two races may differ in aptitudes for certain lines of endeavor, but their value

to society may be equal. It is only when a comparison of the value of the sum-total of contributions to civilization of one race has been found in a great measure less than that of the other, as would be in the case of the Australian, for instance, and any of the European races, that we may use the terms inferior or superior. Keeping this in mind, we believe that on the whole in the case of the Jew, intellectually he is neither superior nor inferior to any of the European peoples. The Jews excel in some lines and fall short in others, and so with the other races. On the whole they compare pretty well. This is borne out by Jacobs' in his study of the "Distribution of Jewish Ability," showing the comparison per mileage of celebrities of Jews with Europeans. We reproduce it in full:

	EUROPEANS	JEWS
Actors	21	34
Agriculture	2	00
Antiquaries	23	26
Architects	6	6
Artists	40	34
Authors	316	223
Divines	130	105
Engineers	13	9
Engravers	3	9
Lawyers	44	40
Medicals	31	49
Merchants	12	43
Military	56	6
Miscellaneous	4	3
Metaphysics	2	18
Musicians	11	71
Natural Science	22	25
Naval	12	25

⁷ J. Jacobs, "The Distribution of Jewish Ability," *Jour. Anthropol. Inst.*, Vol. XV, pp. 351-379.

	EUROPEANS	JEWS
Philologists	13	123
Poets	20	36
Political Economy	20	26
Science	51	52
Sculptors	10	12
Sovereigns	21	12
Statesmen	125	83
Travelers	25	12

This table shows a preponderance of Jewish excellence as actors, doctors, financiers, philosophers, musicians, philologists, poets, a slight excess as antiquarians, in natural science and political economy. They are below in agriculture, novel writing, divinity, engraving, military and naval science, as sovereigns, statesmen and travelers; slightly below as painters, engineers and lawyers. They are about equal as architects, scientists and sculptors. Of course some allowance must be made for the fact that the great bulk of Russian Jewry is practically barred from obtaining eminence on account of political and social oppression, as are also German Jews from entering naval and military professions as well as from statesmanship. It is also seen from this table that Jewish ability tends more in the line of abstract thought, which is partly doubtless due, as pointed out by Jacobs, "to their long life in cities and exclusion from nature on the one side, and from education which lies in handicrafts, on the other."⁸

If we class military and naval under one head, as also sovereigns and statesmen, since they are interdependent, we see that the Jews greatly excel in 7 subjects and are below in 7; they slightly excel in 3 and are slightly below in the same number; they are equal in the others, so that both sides compare equally well. Of course there is another question as to whether the same value is to be attached

⁸ *Loc. cit.*

to the different subjects. Should we, for example, rate equally military science and philology, or agriculture and music, or philosophy and statesmanship? But I think we can easily dismiss this difficulty if we only bear in mind that it is all these combined that make up civilization and all are necessary and important links. Considering this, we can, I think, without reserve accord the Jew a place in higher civilization equal to that of any of the so-called Aryan stock.

We come to the second point: Is the Jew a Semite or an Aryan? We can easily dismiss this by simply remarking that the original composition of the Jew is absolutely of no consequence whatsoever. What matters it whether the Jews four or five thousand years ago were Hittites, Amorites, Semites, or a conglomeration of them all? It is not what entered into their make-up, but what they are *now* that is of importance, and what they are now they are by virtue of a long history and specific phylogeny, the only things that make and create races.

And now as to the first question. Have the Jews a right to be considered as a unity, call it race, people, nation or what not? Or are they simply a heterogeneous mass with no coherence or common elements, as Fishberg's⁹ arguments would imply? We must note in the first place that the effect of environment on variation of type will be greatest with the Jews, on account of their scattered condition and frequent wanderings, change of habitat, abnormal social and economic conditions, and so forth. Concerning the effect of environment, an authority like Beddoe is inclined to believe that both pigmentation and the form of the skull are directly influenced by the kind and quality of food, apart from its sufficiency or insufficiency in quantity. Robert Gordon Latham thought that form and color might in some degree depend on the geological structure

⁹ M. Fishberg, *The Jews*, 1911.

of the habitat, the abundance of carboniferous limestone favoring development of form and complexion. Durand de Gros finds physical differences between the people of the calcareous and granitic parts of Rouergue (south of France), which he thought cannot be accounted for by difference of race. The inhabitants of the calcareous parts are of better form and complexion, while those living in the granitic country are smaller, inferior in form and complexion, less strong but more active. Excess of phosphate of lime in food seems to conduce to good physical development. Thus in Nidwalden and Ticino, two cantons in Switzerland, are found the most robust men, owing, in Beddoe's opinion, to only one point which they have in common, the consumption of great quantities of cheese, an aliment exceedingly rich in phosphate of lime.¹⁰ Again, Professor Lyde points out that pigmentation is not alone influenced by temperature but also by the amount of humidity in the air, the latter favoring fairness.¹¹ Sergi believes that the presence of blondness in North Africa, which has been advanced as an argument against the effect of environment, is to be attributed to the influence of altitude. Its center of formation was in the Atlas valleys, especially Morocco, which is a region of perpetual snow and cold, not unlike some Alpine and Apennine valleys. From there he thinks it has spread into the neighboring countries as far as the sea in Algeria and Tunis. Ridolfo Livi finds that in Piedmont, Liguria, Veneto, Emilia, Lombardy, Tuscany, Marches, Lazio, Campania, Basilicata, Calabria, Sicily and Sardinia, beyond 401 meters above sea level the blonds predominate over the brunettes, with the exception of Umbria and Abruzzi. The exception he attributes to the fact that those two provinces are hilly almost throughout, with

¹⁰ J. Beddoe, *The Anthropological History of Europe*, 1912, pp. 34-36.

¹¹ L. W. Lyde, *Climatic Control of Skin Color*, 1911, pp. 104-108; Spiller, G. (ed.), *Inter-Racial Problems*.

no marked difference between the small plain regions and the surrounding hills.¹²

Indeed, if these arguments bear any weight at all toward the explanation of fairness and darkness in general, their importance should be greater with regard to the Jews, who have been subjected to all climes and all conditions. The fact that the Jews resemble closely the peoples with whom they live, as is seen from the table given below, confirms rather than disproves the theory of climate. This has been conclusively proven by Boas, who has shown that there is a decided tendency in the offspring of immigrants to approach the native head-form. Surely mixture would not account for this change. The explanation is simple; the aborigines or the first settlers of any country have their head-form shaped by the climate and habitat, and any people migrating into the same country undergo the same change without necessarily mingling in blood. The effect on pigmentation may be the same, but the change is so much slower that it becomes perceptible only after millennia.

Following is a table taken from Fishberg¹³ showing comparison of cephalic indices of Jews and their non-Jewish neighbors.

COUNTRY	AVERAGE OF CEPHALIC INDEX OF	
	JEWS	NON-JEWS
Lithuania	81.05	81.88
Roumania	81.82	82.91
Hungary	82.45	81.40
Poland	81.91	82.13
Little Russia	82.45	82.31
Galicia	83.33	84.40

The differences as seen from this table are slight, being greatest in Roumania and Galicia, where it exceeds only

¹² G. Sergi, *The Mediterranean Race*, 1901, pp. 73-75.

¹³ M. Fishberg, *The Jews*, 1911, p. 52; F. Boas, "Changes of Bodily Form of Immigr. Desc.," *Abst. of Reports of Immigr. Comm.*, Vol. II, 1911, pp. 501-556.

one unit. As intermarriages in these countries are least likely to occur, the probability of the effect of environment in tending to approximate the native head-form is still increased.

But environment is not the only factor that may explain the presence of blonds. Even heredity points that way. Experiments in the inheritance of color tend to show that whereas offspring as a rule do not exceed their parents in intensity of pigmentation, they frequently are of a lighter color, so that darker parents may produce light offspring. Davenport, on investigating the inheritance of hair-color in man, finds that out of 210 children whose parents had black hair 3 had flaxen hair, 4 yellow, 5 yellowish-brown, 8 golden, 60 light brown, 37 brown, 49 dark brown, 40 black, and 4 red. It is seen from this that 156 or fully 74 percent of the total had hair lighter than their parents. Davenport also investigated inheritance of eye-color and hair-form, and combining the results of the three investigations he concludes: "It appears that two parents with clear blue eyes and yellow or flaxen straight hair can have children only of the same type, no matter what the grandparental characteristics were; that dark-eyed and haired, curly-haired parents may have children like themselves, but also of the less developed condition."¹⁴

Of course it may be argued that these are the results of segregation or alternate inheritance in the F₂ generation in the Mendelian sense, but his expectations do not exactly tally with his results and are far from being precise, which he himself admits. It is more likely that the results are due to a slight suppression of the pigment factor, the cause of which may be physiological. In the cases of three plants—the sweet pea, the stock and the orchid—Bateson finds that the production of color depends upon a fortuitous con-

¹⁴ C. G. and G. B. Davenport, "Heredity of Hair Form in Man," *Amer. Nat.*, Vol. XLII, 1908; "Heredity of Eye Color in Man," *Science*, N. S., Vol. XXVI, 1907; "Heredity of Hair Color in Man," *Amer. Nat.*, Vol. XLIII, 1909.

course of complementary factors which are independently distributed in gametogenesis, and individuals lacking either of these factors are entirely devoid of color.¹⁵ In the same way it is possible, if one of these factors is partially suppressed by the influence of some external cause, that colors of a lesser degree of pigmentation will arise.

Lightness of color in offspring, unlike parents, may also be due to variations or mutations in the De Vries sense, not of course resulting as he thinks in the creation of an entirely new type, but in the creation of a new character. Brachycephalism among Jews may be due, as pointed out by Jacobs, to intense mental activity, greater among the Jews than any other people.

No less probable is Salaman's suggestion that the divergence in type may be due to the union of characters in gametogenesis in a way similar to that of Bateson's peas, where two apparently similar white sweet peas when mated together gave rise to a purple pea, and when the latter was interbred it produced a series of purples, reds and whites.¹⁶

Quite probably, also, blondness among Jews is to be attributed, as Von Luschan and Haupt are inclined to believe, to the original constituents of the Hebrews, the Hittites and Amorites. The objection that Fishberg raises that in that case the proportion of blonds among Jews in all parts of the world would be the same¹⁷ does not seem to us to hold, for it may be due to the unequal distribution of the blond elements, so that one place may have more and another place less than it should have in proportion to the total number of Jews in that place, aside from environmental and other factors that may produce disproportion.

But, on the other hand, if we even admit that mixture is the only cause of diversity of types among Jews, it could

¹⁵ W. Bateson, *Mendel's Principles of Heredity*, 1913, pp. 88-97.

¹⁶ R. N. Salaman, "Heredity and the Jew," *Jour. of Genetics*, Vol. I, 1910-11, pp. 273-290.

¹⁷ M. Fishberg, *The Jews*, 1911, p. 507.

hardly be explained, it seems to me, on the basis of Mendelian segregation, for since less pigmentation is usually recessive to more intense pigmentation, then in the mating of Jew and non-Jew the former will be dominant and the latter recessive as regards color of eyes and hair. Using the Mendelian formula¹⁸ we would have this:

$$\begin{array}{ll} \text{DD} \times \text{RR} \text{ gives all} & \text{DR} \\ \text{DR} \times \text{RR} & \text{ " IDR : IRR} \\ \text{DR} \times \text{DD} & \text{ " IDD : IDR} \\ \text{DR} \times \text{DR} & \text{ " IDD : 2DR : IRR} \end{array}$$

We must add by way of information for the enlightenment of the general reader that the terms "dominant" and "recessive" as used in Mendelian literature designate the degree of manifestation of one or the other of the individual parental characters in the offspring of two crossed varieties or species, commonly known as hybrid. Hence any character such as size, form, color, etc., which is transmitted entire or almost unchanged in hybridization is termed "dominant," and that which becomes latent in the process "recessive," the latter meaning that the character has either withdrawn or entirely disappeared in the hybrid but may nevertheless reappear again in their progeny. The symbols used to express the relationship of any two pairs of characters are DD "dominant" and RR "recessive" and their combinations, while F₁ denotes first hybrid generation, F₂ second hybrid generation, and so on.

With this in mind, analyzing the above formulas we see the F₁ generation will all appear dominant, in this case of the color of Jewish hair and eyes. When F₁ marries again non-Jewish we shall expect the offspring equally divided between light and dark, but we must note that in this case where the non-Jewish marriage occurred for two generations in succession the third generation, which

¹⁸ W. Bateson, *Mendel's Principles of Heredity*, 1913, p. 12.

should contain blond hair and light eyes, will be in the non-Jewish fold. If we take another possible combination, that of $DR \times DD$, in this case hybrid and Jew, the result will be $1DD:1DR$, or all the offspring appearing dark, so that even if the second generation should marry Jewish and become a member of the community the hair and eyes of the third generation will still appear Jewish and the type of the Jews unchanged. If we take the third combination, where two hybrids intermarry, we shall have non-Jewish color of hair and eyes appearing only in the proportion of $1:3$, but the question is in the first place whether hybrids marrying *inter se* will turn to the Jewish or to the non-Jewish fold, very likely to the latter; in the second place, marriages of hybrids of Jews and non-Jews are least likely to occur, owing, as Salaman pointed out, to the greater choice the hybrid has in finding his mate either in the Jewish community or outside of it. He himself in testing the heredity of the Jewish expression by the Mendelian principle could not find a single example of hybrid mating with hybrid.¹⁹ It is clear, therefore, that the hypothesis of mixture as an explanation for the presence of blond hair and blue eyes among the Jews entirely fails when considered in the Mendelian sense. Surely the number of such cases would be if not *nil*, at least so small that it could produce no perceptible change.

But let us not forget that the problem of heredity of color in man is far from being settled, aside from other considerations, because of the complexity of the transmission of the various color characters. Even Bateson points out that only the inheritance of eye-color alone has been established with any clearness, but with respect to hair-color nothing can yet be said with confidence.²⁰ The task is much more difficult in the intercrossing of races.

¹⁹ R. N. Salaman, "Heredity and the Jew", *Jour. of Genetics*, Vol. I, pp. 273-290.

²⁰ W. Bateson, *Mendel's Principles of Heredity*, Cambridge, 1913, p. 205.

Indeed, so many are the factors involved in the inheritance of characteristics in man that no one factor, and the least of all mixtures, can be taken as the only cause. Brinton believes that the variability of traits within the racial limits is an ethnic principle, and that this becomes greater as the race is higher in the scale of organic development. To quote: "No race remains closer to its type than the Austrafrican, none departs from it so constantly as the Eurafrican. Wherever we find the unmixed white race we find its blond and brunette varieties, its prognathic and orthognathic jaws, its long-skulled and broad-skulled heads. To establish genealogic schemes exclusively on their differences, as has been the work of so many living anthropologists, is to build houses of cards."

Researches conducted by Virchow, De Candolle, Kollman and others disclosed the fact that in the same city and the same family the children are born brunettes or blonds, dark or light eyed, and to some degree broad or narrow skulled, regardless of their parents' peculiarities.²¹ Indeed the writer himself can testify from his own observations, perhaps taking himself as an example, of cases who are of pure Jewish descent, and who can trace their ancestry back for several generations, and who not alone have blond or brown hair but present various ethnic traits in various combinations. But on the other hand the Jews after all are not entirely devoid of common physical characters: they are certainly no more heterogeneous as regards head-form and complexion, the only characters that can be relied on safely in anthropology, than any of the other European races if we except the Jews of Cochin China, the Falashas of Abyssinia and the Samaritans, who in our opinion should not be classed as Jews. Historically the Samaritans have never been part and parcel of the Jewish people; they have not undergone the same shaping and

²¹ D. G. Brinton, *Races and Peoples*, 1890, pp. 108-109.

moulding under the same rod by the same forces that have made the Jew as we see him in Europe to-day, even though we admit that they are of common origin, since it is not the genetic but the developmental factors that create a race; not what it was, but what it is. A belief in the Jewish religion alone does not by any means make one a Jew, any more than a negro would be reckoned as belonging to the Anglo-Saxon or Teutonic race because like them he believes in Christianity. It is, besides, a question whether even by origin they could be classed as Jews. Peschel in a footnote emphatically states that the black Jews of Cochin are natives of India, purchased as slaves by true white Jews, and received into the community after the fulfilment of the Mosaic rites.²² Rohlfs, cited by Jacobs, denies Jewish features even to the Falashas; they are only a negroid element converted to Judaism.²³ The Samaritans are a hybrid people of Jews, Moabites and Amorites, but owing to their complete geographical isolation and practical non-mingling with the other Jews they have not shared in the historical process with the bulk of the other Jews, and cannot properly from a scientific point of view be included in the Jewish race. The same applies to the Karaites, the Daggatauns of the Sahara, the Beni Israel of Bombay, and other tribes in China and elsewhere, which can be reckoned only as religious sects, adhering to the tenets of the Hebrew religion, but not forming part of the Jewish race. The Jews that constitute the Jewish race are those of Europe, Asia Minor and North Africa, and especially those of Russian Poland, Austria and Germany, and the United States, and if we confine ourselves to these, as we should, we shall presently see that they present remarkable uniformity in headform and complexion.

²² O. Peschel, *The Races of Men*, 1906, p. 11.

²³ J. Jacobs, "On the Racial Characteristics of Modern Jews", *Jour. of Anthropol. Inst.*, Vol. XV, 1886, p. 43.

The following table compiled by Ripley²⁴ gives the cephalic indices as found by various investigators at different times:

AUTHORITY	PLACE	NUMBER	CEPH. INDEX
Lombroso (1894)	Turin, Italy	112	82.0
Weisbach (1877)	Balkan States	19	82.2
Majer and Kopernicki (1877)	Galicia	316	83.6
Blechmann (1882)	W. Russia	100	83.21
Stieda (1883)	Minsk, Russia	67	82.2
Ikoff (1884)	Russia	120	83.2
Majer and Kopernicki (1885)	Galicia	100	81.7
Jacobs (1890)	England	363	80.0
Jacobs (1890)	England (Sephardim) ..	51	
Talko-Hryncewicz (1892)	Lithuania	713	
Deniker (1898)	Caucasia	53	85.2
Weissenberg (1895)	S. Russia	100	82.5
Weissenberg (1895)	S. Russia	50 (women)	82.4
Gluck (1896)	Bosnia (Spagnoli) ..	55	80.1
Livi (1896)	Italy	34	81.6
Elkind (1897)	Poland	325 { (men)	81.9
		(women)	82.9
Deniker (1898)	Daghestan	19	87.0
Ammon (1899)	Baden	207	83.5
Ikoff (1884)	Constantinople	17	74.5

The cephalic indices as seen from this table taken at random among Jews of various countries range from 80 to 83, with the exception of Caucasia, Daghestan and Constantinople, being greatest in Daghestan and smallest in Constantinople, although we cannot attach much weight to these extreme cases, since there the number of observations are so few. From this we excluded Ikoff's observations on 30 Caraims in Crimea with a cephalic index of 83.3, who as we said before cannot be classed properly with the Jews. But what is remarkable is the fact that the observations in Russia, Galicia, Poland, Italy and Baden present the least differences, not exceeding two units which may well be attributed to individual variation. Of all these only .08 are dolichocephalic, while all the rest (fully 99.92 percent) are brachycephalic.

The greatest argument against uniformity of skull is

²⁴ W. Z. Ripley, *The Races of Europe*, 1899, pp. 368-400.

based on the assumption that the Sephardic Jews, as distinguished from the Ashkenazim, are dolichocephalic. This has never been founded on facts, for the observations made are exceedingly few; but what is more, from such data as is available, even among them the majority are brachycephalic. This is seen from the above table in the case of the Jews from Bosnia and Italy. Jacobs in London finds among the Sephardim about 11 percent even less pure long-headed than among Ashkenazim.²⁵ Ikoff is the only one who found Sephardim dolichocephalic, but since he observed only 17 crania, no weight can be attached to his results.

Von Luschan made measurements of 1222 Jews, 52 percent of whom were Sephardim of Smyrna, Constantinople, Makri and Rhodes, while the rest were Ashkenazim from Vienna, Austria.²⁶ Unfortunately he does not give the numbers and indices corresponding to each, but from his curve we find only 47 out of a total of 244, or 19 percent, are dolichocephalic, only 33, or 13 percent, are mesocephalic, while the remaining 68 percent are brachycephalic. Of course we do not know how many of the Sephardim were actually brachycephalic, but the exceedingly small percentage of dolichocephals makes it probable that the majority were brachycephalic. Besides, his curve is faulty in that it contains only one-fifth of the actual number, and we are inclined to think that the author picked out only those that show great variance in head-form, in order to prove the extreme variability of the head-form among Jews, a point which he is trying to bring out. We have no doubt the curve would have been different had the total number been plotted, but even as it is it shows up favorably the other way.

²⁵ J. Jacobs, "On The Racial Characteristics of Modern Jews", *Jour. of Anthropol. Inst.*, 1886, pp. 23-63.

²⁶ F. von Luschan, "The Early Inhabitants of Western Asia", *Jour. of the Anthr. Inst. Gr. Brit. and Ire.*, pp. 221-244.

The same uniformity is to be seen from the following additional figures obtained by other observers:²⁸

PLACE	NUMBER	OBSERVER
U. S. Immigr. from Galicia	83.33	Fishberg
U. S. Immigr. from S. Russia . .	82.45	Fishberg
W. Russia	81.05	Fishberg
England	80.00	Jacobs
U. S. Immigr. from Poland	81.91	Fishberg
U. S. Immigr. from Roumania .	81.82	Fishberg
U. S. Immigr. from Hungary . .	82.45	Fishberg
U. S.	81.05	Fishberg
U. S. Immigr. from Persia	81.77	Fishberg
U. S. Immigr.	83.00	Boas
U. S.	81.4	Boas

The difference in all these does not exceed 2, with the exception of England which shows a difference of 3. What is rather remarkable is the exceeding uniformity of all the immigrant Jews in this country, the difference being less than two.

Turning to complexion we find that the brunette type is prevalent, the blonds not exceeding 30 percent anywhere, and being doubtless a result of individual variation. Thus Majer and Kopernicki in Galicia, cited by Ripley, found dark hair to be about twice as frequent as light. Elkind, in Warsaw, finds about three-fifths of the men dark. In Bosnia, Glück found only 2 light-haired men out of 55. In Germany pure brunette types are three times as frequent as light, while in Austria they are twice as frequent among Jewish children as among Christian.²⁹ Of 60,000 Jewish schoolchildren examined in the latter country only 27 percent had blond hair. In Hungary 24 percent of Jewish children had fair hair, in Bulgaria 22 percent. Of 600

²⁸ M. Fishberg, *Die Rassenmarkmale der Juden*, 1913, p. 29.

²⁹ W. Z. Ripley, *The Races of Europe*, 1899, p. 391.

children examined by Fishberg in the schools of the Alliance Israelite in Algiers, Constantine and Tunis only 6 percent had fair hair. Among 4235 Jews observed by the same author³⁰ in New York the following proportions were found:

	JEWS	JEWESSES
Brunette type	52.62%	56.94%
Blond type	10.42%	10.27%
Mixed types	36.96%	32.79%

This table shows only 10.42 percent pure blonds. In the mixed types are included those who have dark hair with fair eyes or *vice versa*, among whom a large percentage must have been of dark complexion. We must bear in mind that a large majority of children become darker in complexion with growing age. Fishberg also finds in North Africa only 4.62 percent of pure blonds. In Bulgaria Wateff found only 8.71 percent blonds. In Austria again, according to districts, Schimmer found only 8 to 14 percent blonds. "Altogether," in Fishberg's words, "it appears that the proportion of Jews of pure blond type oscillates between 5 and 16 percent, according to the country of birth."³¹

Nor is there any striking difference in complexion between the Ashkenazim and the Sephardim. Jacobs³² gives the following data:

(a)		LIGHT EYES			NEUTRAL EYES	DARK EYES	
		290	Sephardim	20	12	68	
	375	Ashkenazim	27	14	59		
(b)		RED HAIR	FAIR HAIR	BROWN HAIR	DARK HAIR	BLACK HAIR	
		290 Seph. . .	3.5	3.5	15.7	40.0	37.3
	375 Ashk. . .	I. I	2.6	17.0	45.6	32.7	

³⁰ M. Fishberg, *The Jews*, 1911, pp. 63-66.

³¹ *Ibid.*, pp. 66-68.

³² J. Jacobs, "On the Racial Characteristics of Modern Jews," *Jour. of Anthropol. Inst.*, 1886, pp. 23-63.

The only marked difference between the two, as seen from this table, is in the frequency of erythrism, which is about three times as frequent among the Sephardim, but the percentage however appear to be large, due to the small number observed. If we combine the brown, dark and black, all of which should really be classed as brunettes, and also the red and fair as blonds, we have 93 percent of brunettes among Sephardim against 95.3 percent among Ashkenazim. The eyes show greater difference, but no definite correlation has been established between hair and eyes, and we think that the hair only can be relied upon to designate complexion. Of course the differences vary in different countries, but what is significant is the fact that the dark type is prevalent in both the Ashkenazim and Sephardim.

The prevalence of dark complexion is also borne out by another table taken from Fishberg,³³ which shows the percentage of dark and fair hair among 2272 Jews of New York City.

COLOR	JEWS	JEWESSES
Dark Hair	83.49%	80.17%
Fair Hair	13.98%	16.14%
Red Hair	2.53%	3.69%

Thus we see that even from the strictly anthropological view-point the heterogeneity of type among the Jews is quite small, certainly not enough to ascribe it to mixture, and certainly less than among other peoples. Shall we say that the Teutons, for example, are less heterogeneous, comprising as they do the Saxons and Hanoverians in the north, who speak *plattdeutsch*; the Netherlanders and Flemings of the north of Belgium, who speak Flemish or Dutch; the southern Germans; the Alemanni of German Switzer-

³³ M. Fishberg, "Phys. Anthropol. of the Jews," *Amer. Anthropol.* N. S. 5, 1903, pp. 89-106.

land, Alsace and Baden; the Swabians of Württemberg and Bavaria; the Bavarians of eastern Bavaria and of Austria, who speak *hochdeutsch*; the inhabitants of middle Germany, the Thuringians, Franconians, etc., who speak *mitteldeutsch*; finally, the Prussians, partly composed of Germanized Slavo-Lithuanian elements?

The same is true of the Slavs, among whom, in Deniker's words, "it is useless to look for a 'Slav Type.'" In the east we have the Great, Little and White Russians; in the west the Poles of Russian Poland, western Galicia, Posen and eastern Prussia, the Wends or Sorobes of Saxony and the Prussian province of Saxony, who are undergoing a process of Germanization; the Bohemians of Bohemia and a part of Moravia; the Slovaks of Moravia and Hungary. In the south there are the Slovenes or Slovintsi of Austria-Hungary, the Khorvates of Hungary, the Serbs of Servia, the Morlacks, etc., of Dalmatia; the Herzegovinians, Bosnians, Montenegrins or Tsarnagortsi in other parts of the Balkan peninsula; and finally the Bulgarians, who are of Turco-Finnish origin, but Slavonized for at least ten centuries. And so are all the other European and Asiatic peoples.³⁴

But more important than physical characteristics are the physiological, pathological and psychological, which are common to the Jewish people as a whole. Of these we can only mention a few. Thus Jacobs in his studies of Jewish biostatics comes to the following conclusions:

1. "Jews have a less marriage rate, less birth rate, and less death rate than their neighbors, but the less marriage and birth rate are due in large measure to the less mortality of Jewish children. The larger number of children living causes the percentage of marriages and births, really larger as regards adults, to seem smaller when reckoned on the whole population."

³⁴ J. Deniker, *The Races of Man*, 1900, pp. 339-348.

2. "Jews and Jewesses marry earlier than the surrounding population. Cousins inter-marry more frequently, perhaps three times as often."

3. "Jews have larger families, though fewer plural births. On the other hand, mixed marriages between Jews and persons of other races are comparatively infertile."

4. "In Jewish confinements there are more boys, less still-births, and fewer illegitimate births, though the advantage as to still-births disappears among Jewish illegitimate children."

5. "Jews have a smaller mortality of children under five, but this does not hold of Jewish illegitimate children, who die off at much the same high rate as the unfortunate beings of the same class in other sects. Jewish deaths over sixty are generally greater in proportion. Jews commit suicide less frequently."

6. "It has been frequently asserted that Jews enjoy an immunity from certain diseases, notably phthisis and cholera, but the evidence I have on this point is adverse to the claims. There is some indication that they are liable to diabetes and haemorrhoids, and they have certainly more insane, deaf-mutes, blind, and color-blind persons."³⁵

The same results were found by Hoffman, Kolb Bergmann, Legoyt, Dernouilli, Lagneau, Loeb and many others.

Lucien Wolf and Dr. Asher who had several years' experience as surgeons to the Jewish Board of Guardians, affirm Jewish immunity from phthisis. Dr. Asher states that in his experience phthisis among English Jews is unknown. This is substantiated by the statistics in the report for 1859 of Dr. Septimus Gibbon, Medical Officer of Health.³⁶

Venereal diseases have been found less frequent among

³⁵ J. Jacobs, "On the Racial Characteristics of Modern Jews," *Jour. of Anthropol. Inst.*, Vol. XV, 1886, pp. 26-27.

³⁶ *Ibid.*, pp. 56-61.

Jews. Dr. A. Cohen, late Senior House Surgeon of the Metropolitan Free Hospital, London, gives the following figures.³⁷

	MEN		WOMEN		CHILDREN	
	NO.	PERCENT	NO.	PERCENT	NO.	PERCENT
Jews	122	17.8	10	20.0	153	3.3
Others . . .	539	62.0	192	62.6	367	15.8

The percentage of the first two rubrics are those of syphilitic cases, the next two are those of gonorrhea, and the last two are those of congenital syphilis observed in the number of children examined.

Cancer is less frequent among Jews. Jacques Bertillon, cited by Fishberg, gives the following figures showing the deaths from cancer per hundred thousand population for different peoples during the years 1903-1908 in Algiers.

French (native)	40
French (naturalized)	18
Jews (naturalized)	21
Spaniards	33
Italians	38

The figures for Amsterdam given by Dr. J. J. Von Konijnenburg are of the same nature.³⁸ Thus:

	MEN	WOMEN
Jews (1898-1902)	60	77
Gen. Population . . . (1897-1902)	90	98

In London during 1898-1900 the proportion per hundred of deaths from cancer to deaths due to all causes among persons over twenty years of age was as follows:

	1900	1899	1898
Jews	6.1	6.5	5.02
General Population . . .	8.4	8.8	6.1

³⁷ *Ibid.*, pp. 31-32.

³⁸ M. Fishberg, *Die Rassenmerkmale der Juden*, 1913, pp. 117-118.

Jewish children have everywhere been found to suffer less from diseases of the digestive organs. Thus, in Budapest, Körösi finds the death rate from infantile diarrhea during the period 1860-90 to have been as follows per hundred thousand children under five years of age:

Catholics	4143
Lutherans	3762
Calvinists	3293
Other Protestants	3498
Jews	1442

In Vienna Rosenfeld finds the mortality of Jewish children from diarrhea to be only 61 per one hundred thousand population, as opposed to 137 of Protestants and 186 Catholics. In New York, Fishberg³⁹ calculated from the reports of the Department of Health that during 1897-99 the annual mortality from diarrhea diseases in the entire city was 125.54 per one hundred thousand population, while in the most congested districts, largely inhabited by Jews, it was only 106.79.

The Jew has been found to be deficient in stature, breadth of chest, and lung capacity, by Jacobs, Majer and Kopernicki, Stieda, Glück and others, but in spite of that his tenacity for life has been unprecedented. Especially is this true in the United States. This may be shown by comparing the vital statistics of the Jews as elaborated by Billings in the census of 1890, with that of the general population.⁴⁰ It is also seen from the following table given by Hoffman,⁴¹ showing the death rate per one thousand population in the seventh, tenth and thirteenth wards of New York City, 1890, by place of birth:

³⁹ M. Fishberg, *The Jews*, 1911, pp. 306-312.

⁴⁰ D. G. Brinton, *Races and Peoples*, 1890.

⁴¹ W. Z. Ripley, *The Races of Europe*, 1899, p. 384.

AGES	TOTAL DEATHS PER 1000	U. S. INCLUDING COLORED	IRELAND	GERMANY	RUSSIA AND POLAND (MOSTLY JEWS)
Total	26.25	45.18	36.04	22.14	16.71
Under 15 years	41.28	62.25	40.71	30.38	32.31
15 to 25 years.	7.55	9.43	15.15	7.14.	2.53
25 to 65 years.	21.64	25.92	39.51	21.20	7.99
65 and over . . .	104.72	105.96	120.92	88.51	84.51

As seen from this table mortality among Jews is a little less than one-third of the general population, a little less than one-half for persons under 15, about one-fourth for ages 15 to 25, less than one-fourth for ages 25 to 65, only slightly less however for ages 65 and over.

Considering Jewish biostatics as a whole it would seem to indicate that the Jews are physiologically and pathologically superior to their neighbors, except as regards insanity, deaf-mutism and blindness, but even in these Billings's figures show up favorably. Thus, for example, the proportions per one hundred thousand of population of insane and idiots reported among Jews is 44.5, while that reported by the United States census of 1880 was 336.6, and by the Massachusetts census of 1885 it was 355.3. For deaf-mutes among Jews the proportion was 31.3 per one thousand as opposed to 67.5 by the tenth census of the United States, and 42.6 by the Massachusetts census of 1885.⁴² This is not however borne out by all the European figures, and may be due, as Billings suggests, to incorrect reports.

Of course there are differences as to locality, and some investigators confining themselves entirely to one locality are inclined to deny or add one or the other characteristic,

⁴² J. S. Billings, *Vital Statistics of the Jews in the U. S.*, Vol. I, No. 19, pp. 1-23.

but by far the greater number of the reports show these differences to be characteristic. Whatever the causes may be, whether dietary laws, family hygiene, beautiful home life, or a result of a long process of selection, the fact remains that those characters are met with in practically every Jewish community.

Intellectually, we have also seen in the beginning of this essay how Jewish intellect tends in a particular direction. This doubtless may be a result, as Lazare points out, of a long continued study of the Torah and Talmud, which shaped the Jewish brain and gave it a characteristic type.⁴³

And finally we come to what we consider the most important factor, namely the psychic personality of the race. We have marshalled up all this evidence thus far, and argued both positively and negatively, in order to show that no matter from what angle you approach the problem, whether environmental, hereditary or physiological, the arguments are in favor of the comparative purity of the Jewish race. By this we do not mean that the Jews have abstained from intermarriage, but rather that, by the nature of the facts, the Jew not being of the dominant race and considered more or less a stranger, the majority of those that intermarried left the Jewish fold forever, and the exceedingly small percentage that remained in the community could not possibly affect the Jewish type, certainly not to any noticeable extent. But let us reiterate. After all, of what import is mixture or non-mixture? Ethnically there is no pure race. The old polygenistic view has long been abandoned by men of science. It is conceded by anthropologists that the modern races have not sprung up independently, but have had a common origin. It is not the origin however but the phylogenetic development that a group of individuals, irrespective of its primary constituents, undergoes that finally moulds it into a distinct

⁴³ B. Lazare, *Antisemitism,—Its History and Causes*, 1903, p. 256.

unit, or what we commonly call race. It is the complete assimilation and fusion of the constituents as a result of long periods of in-breeding and subjection to similar conditions and customs that makes the race. The *summum bonum* of the phylogeneticism is the psychic personality, the soul or race consciousness, if you choose, of each race; and if this is true of any people it is especially true of the Jews, who have tenaciously displayed it in the face of all opposition, with no political boundaries and no center of their own. The characteristic Jewish expression, which even Ripley, Fishberg and Weissenberg do not deny, is, as Fishberg thinks, "the expression of the Jewish soul";⁴⁴ but, unlike him, we maintain that it is the most potent, determining factor for each and every race, that it is by far the best guide for distinguishing one race from the other; and while physical characters fail, being as they are subject to environment, physiological, and other changes, it persists in spite of all outward changes. That this is so with the Jews is remarkably affirmed by Salaman's study, who found it to Mendelize, and whose results we give here:

FIRST GENERATION

NO. OF FAMILIES	FATHER	MOTHER	GENTILES	CHILDREN		
				JEWS	INTER- MEDIATES	
50	Gentile	Jewess	88	15	4	
86	Jew	Gentile	240	11	4	
Total 136			328	26	8	

Out of a total of 136 families tested, of which 50 consisted of father Gentile and mother Jewess, and 86 father Jewish and mother Gentile, there were 328 offspring of Gentile appearance, 26 Jewish, and 8 intermediates; but since the intermediates were more Gentile-looking than

⁴⁴ M. Fishberg, *The Jews*, 1911, p. 165.

Jewish, the result is 336 Gentile-looking against 26 Jewish, or the ratio of 13.1. The Mendelian expectation which should have given absolute dominance is short by one, which Salaman attributes to the bias of his observers, who, being zealous Jews, may have taken non-Jewish-looking for Jewish, but what is more probable is the possibility that the non-Jewish parent had Jewish blood. This he actually found to be the case in one family, whose pedigree we reproduce here. All the other families refused to give their genealogies. It may also be due to incomplete dominance, which is quite prevalent even among lower animals and in plants.

Of mating of hybrid and hybrid, Salaman could not find any cases, but he tested 13 families, of which 9 were matings between hybrid mother and Jewish father, and 4, hybrid father and Jewish mother. They had a total of 15 children Gentile-looking and 17 Jewish, as is seen from the following table:

NO. OF FAMILIES	FATHER	MOTHER	CHILDREN	
			GENTILE	JEW
9	Jew	Hybrid	13	12
4	Hybrid	Jewess	2	5
Total 13			15	17

The results as seen from the table fall short of expectation only in two cases. In a personal communication with Dr. Salaman he informs us that he has now additional data which bear out the same results, but which he has not published on account of the war. The results show clearly that the Jewish facial expression behaves as a recessive character to the Gentile, but that it is hereditary just the same. On the other hand the non-Jewish appearance frequently met with among Jews was found by

Salaman to behave as recessive to the pure Jewish appearance.⁴⁵

The persistence of the Jewish type is also beautifully illustrated in Galton's composite photograph compounded of a number of photographs of Jewish boys from the Jews' Free School, London.⁴⁶ The typical Jewish expression is remarkably displayed.

It is quite clear that the facial expression of the Jew is a true character, and that therefore the inner psychic personality of the race, of which it is only the outward manifestation, is likewise true and fundamental. The question has been raised as to what has caused the Jewish expression. Some think it is largely a result of long exile and social isolation, as Jacobs suggests; Ripley thinks it is a matter of artificial selection; Fishberg thinks much of it is due to the Jewish costume, etc. But if we keep in mind that the race is the totality of all the elements that have played a part in its history, we can easily see that the expression is a reflection of all the forces that shaped the destiny of the Jewish people. It is neither the result solely of Ghetto life, least likely is it a result of artificial selection, nor can dress and social surroundings change it; they may make it less accentuated, but the features cannot be demolished. In a word, it is not, in our mind, the result of any one thing, but it is a fusion of all the elements that made the Jew as we know him to-day. If we were asked to give those elements we would name them as follows: The sublimity and righteous indignation of the prophets and scribes; the pathos and tragedy of ages of persecution and martyrdom; the cunning and shrewdness that is characteristic of all people who have to live by their wits; a shade of anger or resentment. Finally, we see in the

⁴⁵ R. N. Salaman, "Heredity and the Jews," *Jour. of Genetics*, Vol. I, 1910-11, pp. 273-290.

⁴⁶ J. Jacobs, "On the Racial Characteristics of Modern Jews, *Jour. of Anthropol. Inst.*, Vol. XV, 1886, pp. 23-63.

Jewish expression the calculation, coldness and scanning which so struck Galton, and which we think is a result of long experience in financial operations. All these elements have by long use and repetition fused and become hereditary. The non-uniformity of expression among the different members of the race are due to differences of individual experience.

And now the question will be asked, If the expression persists does it follow that the racial consciousness will likewise persist? We have mentioned before that the expression is only the physical manifestation of the psychic, and we are inclined to believe with Von Luschan, Wirth and others that race consciousness may never disappear. At any rate, what the future holds cannot be prognosed, but the present shows that race consciousness, far from declining, is being enhanced, and this not only among the Christian peoples but among the Jews as well, and no less among American Jewry than among European. We have clear evidence of this in the remarkable progress of the Menorah societies among Jewish students all over the country. Founded in 1906 as a local society at Harvard, it has now spread all over and became intercollegiate with a separate organ of its own. The chancellor calls attention to the fact that within the last two years Menorahs have grown from nineteen to thirty-five, and this without the slightest agitation on the part of those interested in the movement. Of course the actual members make up only a small percentage of the great bulk of Jewish students, but let us not forget that by far the majority of students, if they for some reason or other do not actively participate, are decidedly in full sympathy with the movement. We have come in contact with all kinds of Jewish students, rich and poor, European or American born, first, second and third generations, east and west, in large and small

Jewish communities, and we know that the sentiments are the same all over.

Not only Menorahs but distinctly Jewish organizations are being formed all along among all classes of people. Y. M. H. A's., Herzl, Montefiore, Disraeli, Judea and numerous other clubs and societies are growing at a tremendous rate; needless to mention Jewish philanthropic agencies. True, religion with Israel is decaying, as it is with all other peoples, making place for broader humanistic conceptions, for a religion on earth, but the religion is not the race. True religion in Israel has played perhaps the most important part in the making of the Jewish race, but it cannot function in its unmaking. The Jew remains a Jew with or without the religion. Nihilist, atheist, or agnostic, he is still a Jew in sentiments and spirit. Divided as the Jews are among themselves, they display unexampled solidarity when anything threatens the whole race. Orthodox and reformed, believer and free-thinker, rich and poor alike, all rally together and form a compact solid wall. Even the proselyte, deceiving as he does his own conscience, is no less a Jew in spirit; and the same is true of the assimilator. Strange to say, Fishberg himself, perhaps the staunchest advocate for assimilation whether by preference or some other reason, prefers to pursue his activities in the Jewish fold and even engages in pure Jewish philanthropy. In short, we firmly believe that the race-consciousness, or what we have termed the psychic personality of the race, in a Freudian sense, which alone is its true determiner, is fully alive with the Jew, and if not extinguishable altogether we may be certain that, for good or bad, it will remain so for a long time to come.

In view of all that has been said in this article we believe that, if the privilege, if such it be, to be called a race is given to any people, it should certainly be given to

the Jews, who, unlike any other people, possess all the characteristics that enter in the make-up of races.

To sum up: We have pointed out the confusion that exists as regards the anthropology of the Jew,—the question as to whether the Jews are a race or religious sect, etc., whether they are Semites, and whether they are superior or inferior to the Aryan races. We showed, as regards the latter, that intellectually they are neither inferior nor superior, but that physiologically they are slightly above their neighbors. The second question we dismissed as being irrelevant. As regards the first question we showed that from all points of view,—environmental, hereditary, strictly anthropological and physiological,—the arguments are against the hypothesis of mixture; and finally we showed that irrespective of mixtures, which are of minor importance when taking place at a remote period, the Jew above all presents a distinct psychic unity, which alone we think can be taken as a safe criterion of any race, and that, in view of all that has been said, if any people is entitled to be designated as a race, it is certainly the Jews.

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LOGISTIC AND THE REDUCTION OF MATHEMATICS TO LOGIC.

IN the year 1901 we find in an article by Bertrand Russell:¹ "The nineteenth century which prides itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics.... One of the chiefest triumphs of modern mathematics consists in having discovered what mathematics really is.... Pure mathematics was discovered by Boole in a work which he called *The Laws of Thought*.... His work was concerned with formal logic, and this is the same thing as mathematics."

Also in Keyser's address² we find: "...the two great components of the critical movement, though distinct in origin and following separate paths, are found to converge at last in the thesis: Symbolic Logic is Mathematics, Mathematics is Symbolic Logic, the twain are one."

On the other hand we find Poincaré³ saying after his various successful attacks on logistic: "Logistic has to be made over, and one is none too sure of what can be saved. It is unnecessary to add that only Cantorism and Logistic are meant, true mathematics, those which serve some useful purpose, may continue to develop according to their own principles without paying any attention to the tempests raging without them, and they will pursue step by step

¹ *International Monthly*, 1901.

² *Columbia University Lectures*.

³ *Science et méthode*, p. 206.

their accustomed conquests which are definitive and which they will never need to abandon."

What then is this logistic which made such extravagant claims in 1901 and in 1909 was dead? In order to understand it we must go back to the third century B. C. when Aristotle was developing the study usually called logic. The logic of Aristotle is well enough defined when it is called the logic of classes. A class may be defined in the following terms: Let us suppose that we start with a proposition about some individual, as for example, "8 is an even number," or as another case, "Washington crossed the Delaware." If now we remove the subject and substitute the empty form x , we shall have the statements: " x is an even number, x crossed the Delaware," which are called propositional functions, from analogy to mathematical functions. In this case the functions have but one variable or empty term, x . If we let x run through any given range of objects, the resulting statements will be some true, some false, some senseless. Those that are true or false constitute a list of propositions. For example we may say: "6 is an even number, 9 is an even number, this green apple is an even number," the first a true proposition, the second a false proposition, the third an absurdity. So I might say: "Washington crossed the Delaware, the Hessians crossed the Delaware, the North Pole crossed the Delaware," which are respectively true, false, and absurd, the first two cases being propositions. The propositional function with one variable is called a concept. The individuals that may be put into the empty term (which may be any word of the statement), the variable, and yield true propositions constitute the class of the concept. Thus the class of even numbers consists of a certain endless set or range of individuals, the class of presidents of the United States a certain set of a few individuals, the President of the United States of one individual, and the class of simple

groups of odd order may consist of no individuals at all. The individuals of a class may not be known, for instance the daily temperatures at the North Pole, or the odd perfect numbers. It is practically impossible to ascertain the individuals in the first class, and there may not be any in the second class mentioned. In case it can be shown that a class has no individuals it is called a null-class. It should be noted carefully that the individuals do not define the class, but conversely the class defines the individuals. The same individuals may be defined by one or more classes. Nor is the relation of a member of a class to the class the same as the relation of a subclass to the class. For instance we may discuss the class of numbers which are either multiples of 5 or give a remainder 1 when divided by 5. Now the class of fourth powers of integers are all either divisible by five or give 1 for remainder. Hence the fourth powers constitute a subclass of the first class mentioned. But of any one fourth power, as 81, say, we cannot assert that 81 has the property of being divisible by 5 or of giving a remainder one, and its relation to the class is different from the relation of the subclass to the class. A subclass is said to be included in the class, not to be a member of it. This difference was first pointed out by Peano⁴ and was not known to Aristotle. The two relations are indicated by the symbols \in and ' \subset ', for instance,

Roosevelt \in presidents of the United States,
some square roots ' \subset ' irrationals.

The symbol of a class is the inverted ϵ , 3, for instance

x 3 divisor of 288,

read "the class of divisors of 288." It is evident that a class is not a class of classes, for the latter is a class of propositional functions of one variable, the former a class of individuals.

⁴ *Formulaire de mathématique*, Vol. I.

Aristotle not only studied classes, with schemes for definition and subdivision of classes, but he introduced the syllogism as a means of reasoning. The syllogism is a succession of three statements of the inclusions of classes; in formal statement, Greek letters denoting classes,

$$\alpha \cdot (\cdot \beta, \beta \cdot (\cdot \gamma, \text{ then } \alpha \cdot (\cdot \gamma.$$

For example, Pascal's theorem is true of any conic, every circle is a conic, whence Pascal's theorem is true of every circle. For an individual circle we should have a different type of syllogism, a distinction not noted by Aristotle, namely

$$x \in \alpha, \alpha \cdot (\cdot \beta, \text{ then } x \in \beta.$$

For instance, Pascal's theorem is true of circles, this figure is a circle, thence Pascal's theorem is true for this individual circle.

Logic rested with the Aristotelian development for many centuries, and was supposed to be perfect. The re-generation of the subject has been ascribed to Leibniz, because he hoped to see a universal symbolism which would enable the complete determination of all the consequences of a given set of premises to be easily carried out, just as mathematical formulas enable us to solve large classes of problems. This was his Universal Characteristic. But it was reserved for a later day to bring to light the symbolic logic, and we may pass at once to Boole⁵ and the nineteenth century. We shall find however in the invention of Boole and his successors not the discovery of mathematics but the mathematicising of logic. The mind again devises new forms for its own use, new ideas by which to attack its problems.

Boole used letters to express classes, the conjunction

⁵ *The Mathematical Analysis of Logic*, 1847; *An Investigation of the Laws of Thought*, 1854.

of two letters indicating the largest common subclass, and the formal addition of two letters the smallest common superclass. Then the laws of logic are stated by the formal equations

$a=aa$, (identity); $a+ab=a$, $a(a+b)=a$, (absorption); $ab=ba$, $a+b=b+a$, (commutation); $aa=a$, $a+a=a$, (tautology); $ab=aba$, $a=a(a+b)$, (simplification); $a=ab$, $a=ac$, then $a=abc$, (composition).

He introduced two constants called logical constants, represented by 1 and 0, with the meaning for 1, the minimum superclass of all classes considered, the logical universe; and for 0, the greatest common subclass of all classes, the null-class, or class of impossibilities. It is understood that if a class is considered, the negative of the class is also under consideration, represented by a' . If only one class is considered then $1=a+a'$. If two are considered $1=ab+ab'+a'b+a'b'$, etc. It is evident that

$$1a=a, 1+a=1, 0a=0, 0+a=a.$$

The invention of these notions which seem simple enough now was a great advance over the logic of Aristotle. It suggested for example the use of $1-a$ for a' , with the formulas corresponding to algebra

$$a(1-a)=0, 1=a+(1-a),$$

the laws of contradiction and excluded middle. Any class may be dichotomized in the form

$$x=ax+a'x=abx+ab'x+a'b x+a'b'x=\dots$$

If x is a subclass of a we indicate it by the equations

$$x=ax \text{ or } xa'=0.$$

The syllogism takes the very simple form

$$a=ab, b=bc, \text{ then } a=abbc=abc=ac.$$

We have thus invented a simple algebra which, with the one principle of substitution of any expression for a letter which the letter formally equals, and the reduction of all expressions by the laws of the algebra, enables us to solve easily all the questions of the older logic. Jevons⁶ has stated the rule for doing this very simply: "State all premises as null-classes, construct all necessary subclasses by dichotomy, erase all combinations annulled by the premises, and translate the remaining expressions, by condensation, into the simplest possible equivalent language."

Boole however made a further most important discovery, that there is a nearly perfect analogy between the calculus of classes and the calculus of propositions. That is, we may interpret the symbols used above as representing propositions, under the following conventions. If a is a proposition, a' is the contradictory proposition, ab a proposition equivalent to the joint assertion of a and b , $a+b$ the assertion of either a or b or both, ι a proposition asserting one at least of all the propositions and their contradictories under consideration, and σ a proposition asserting all the propositions and their contradictories simultaneously, that is, ι asserts consistency, σ inconsistency. A series of formal laws may now be written out and interpreted similar to those for classes. The syllogism, for instance, is the same,

$$a=ab, b=bc, \text{ then } a=ac; \text{ or in equivalent forms,}$$

$$ab'=\sigma, bc'=\sigma, \text{ then } ac'=\sigma.$$

That is, if the assertion of a is equivalent to also asserting b , and if the assertion of b is equivalent to also asserting c , then the assertion of a is equivalent to the assertion of c . We may reduce the whole scheme of deduction as before to a system of terms which are the expansions of the possible list of simultaneous assertions, the premises annulling

⁶ *Principles of Science*, also *Pure Logic*. See also *Studies in Deductive Logic*. Also Couturat, *Algèbre de la logique* (*Algebra of Logic*, translated by Robinson).

certain of these, and those remaining furnishing the conclusions. We should however note carefully that what we arrive at in this manner are not truths or falsehoods but consistencies and inconsistencies. That is to say, we do not prove anything to be true or false by the logic of propositions, we merely exhibit the assertions or classes with which it is consistent or compatible, or the reverse. In this sense only does logic furnish proof. It is obvious however that many new combinations of the symbols used are possible by these methods, and thus it is easy to ascertain the consistency of assertions that would not otherwise occur to us. While the premises evidently are the source of the conclusions, the conclusions are not the premises, and on the one hand the transition from the one to the other is made most easily by these methods, and on the other hand the conclusions are new propositions consistent with the premises. A simple example will show what is meant:

If a implies a' , then a is 0; for if $aa=0$, at once $a=0$.

Conv. if $a'a=0$, $a'=0$, $a=1$.

That is, a proposition which implies its contradictory is not consistent.

It should be noted that the calculus of propositions is not wholly parallel to the calculus of classes. This is shown particularly in the application of a certain axiom, as follows:

$(a \in \text{true}) = a$ Ax. $a' = (a' \in \text{true}) = (a \notin f)$. This is absurd for the logic of classes, since $a=1$ is a proposition and not reducible to a class.

A useful form for implication is

$(a \text{ implies } b) = (a' + b = 1)$.

The next advance was due to C. S. Peirce,⁷ who devised

⁷ *Mem. Amer. Acad. Arts and Sciences*, N. S., IX, 1870, pp. 317-378.

the logic of relatives, in which the propositional function with two variables appears, and which may readily be generalized into the propositional function with any number of variables, giving binary, ternary, and then n -ary relatives. As simple examples we may omit individuals that satisfy the proposition: A is the center of the circle c , arriving at the propositional function: x is the center of y ; or another example with four variables is found in: x is the harmonic of y as to u and v . The calculus of the logic of relations is obviously much more complicated than the previously known forms of symbolic logic. While some of the theorems and methods of the calculus of classes and propositions may be carried over to the calculus of relations, there are radical differences. For instance the relation xRy is the converse of the relation yRx . These two relations are not identical unless R is symmetric. Again from xRy , yRz , we can infer xRz only if R is transitive. The ranges of a relation are the sets of individuals that satisfy the propositional function, when inserted for some one of the variables. The most complete development of these notions is to be found in Whitehead and Russell's *Principia Mathematica*. In the intoxication of the moment it was these outbursts of the mind that led Russell into the extravagant assertions he made in 1901. In the *Principia* there are no such claims. It should be noted too that the work of Whitehead in his *Universal Algebra* (1898) contained a considerable exposition of symbolic logic.

As soon as the expansion of logic had taken place Peano undertook to reduce the different branches of mathematics to their foundations and subsequent logical order, the results appearing in his *Formulario*, now in its fifth edition. In the *Principia* the aim is more ambitious, namely to deduce the whole of mathematics from the undefined or assumed logical constants set forth in the beginning. We

must now consider in a little detail this ambitious program and its outcome.

The basal ideas of logistic are to be found in the works of Frege, but in such form that they remained buried till discovered by Russell after he had arrived himself at the invention of the ideas independently. The fundamental idea is that of the notion of function extended to propositions. A propositional function is one in which certain of the words have been replaced by variables or blanks into which any individuals may be fitted. This isolation of the functionality of an assertion from the particular terms to which it is applied is a distinctly mathematical procedure, and entirely in line with the idea of function as used in mathematics. It enabled us above to define concept and relation, in a way, and it further makes quite clear in how great a degree mathematical theorems are about propositional functions and not about individuals. For instance, the statement, "If a triangle has a right angle it may be inscribed in a semicircle," merely means

right-angled-triangularity as a property is inconsistent with non-inscribability-in-a-semicircle as a property.

In this mode of statement it is apparent to every one that a large part of mathematics is concerned with the determination of such consistencies or inconsistencies. That it is not wholly concerned with them however is also quite apparent. For example, the calculation of π can only be called a determination of the figures consistent with certain decimal positions by a violent straining of the English language. And again, the determination of the roots of an equation is a determination of the individuals which will satisfy a given propositional function, and not a determination of the other functions consistent or inconsistent with that first function. There is a difference well known to

any mathematician between the properties of the roots of a quadratic equation and the properties of quadratic functions of x . Again, the analysis of the characteristics of a given ensemble is a determination of the essential constituents of the propositional function whose roots are the individuals of the ensemble. Operators considered as such are not propositional functions, and neither are hyper-numbers. It has been made quite clear, we hope, in what precedes, that much of the mathematician's work consists in building up constructions, and determining their characteristics, and not in considering the functions of which such constructions might be roots. There is a difference between the two assertions

$2 + 3 = 5$ and, If 2 is a number, and if 3 is a number, and if 2 and 3 be added, then we shall produce a number which is 5.

We find the difference well marked in the logistic deduction of the numbers one and two. The deduction is as follows:

Let us consider the propositional functions

" $x \in \varphi_1$ has only roots such that they cannot be distinguished," as likewise $x \in \varphi_2, \dots$ For instance let $() = 6$, of which the roots are $4 + 2$, 2×3 , $12/2, \dots$ which are all indistinguishable in this propositional function. So also $() = 9$, $() = 4/3, \dots$ Then if we call these propositions similar, in that each *has indistinguishable roots*, we may consider next the propositional function $p \sim () = 6$, where p is a variable proposition, which however is distinguished by the character of indistinguishable roots. We may now define the number 1 as the functionality in this functional proposition. That is to say, 1 is a property of propositional functions—namely, that of uniqueness of their roots. In mathematical language we might say: The character which is common to all equations of the form $(x - a)^n = 0$, is called *one*, thus defining *one*. Now while it is true perhaps,

that to seize upon equations with one root as cases in which oneness appears, is a valid way to arrive at *one*, nevertheless it is not at all different from any other case in which oneness occurs, as in selecting one pencil from a pile of pencils. In a like manner two is defined as the common property of propositional functions which are relations with a twofold valence, that is, admit two series of roots, the series in each case consisting of indistinguishable individuals. The truth of the matter is that the definitions given are merely statements in symbolic form of cases in which the number one or the number two appears. The two numbers have in no wise been deduced, any more than a prestidigitator produces a rabbit from an empty hat, but they have first been caught, then simply exhibited in an iron cage. The fact that functions are useful things we cheerfully admit, but that everything is reducible to logical functions we do not admit. The arithmetic of 2 and 1 was known long before logistic.

Another notion introduced by logistic is that of truth and truth-value. In no place are either of these terms made clear, nor are they defined. They are qualities of *propositions*, that is propositional functions which have had individuals inserted for the variables. For example, if I consider the propositional function x is right-angled, and then for x insert respectively the triangle ABC, the parallelogram S, this pink color, I have the propositions ABC is right-angled, the parallelogram S is right-angled, this pink color is right-angled. The first of these is said to have the truth-value *truth*, the second the truth-value *false*, the third has the value *absurd*, which is not a truth-value. The first two assertions are then propositions, the third is not a proposition. Much is made of the idea of truth-value, but practically it amounts only to saying that an assertion is a proposition only when it can be labeled with one of two given labels. If any other label is neces-

sary it is not a proposition and not within the region of logistic. So far as really used in logistic these labels are neither more nor less than labels of consistency and inconsistency. They do not refer in any way to objective truth. Thus if we start with the postulates of Euclidean geometry we arrive at certain propositions, as, "triangle ABC has the sum of its angles equal to two right angles." This proposition is not to be tagged as true, but merely as consistent with the premises we started with. The determination of the primitive truth of the premises is not possible by logistic at all. The whole of science is of this character, the truth of the conclusions of science being only probable, not certain, although the reasoning is valid. Science draws its validity from the agreement of all its conclusions with experience. In the same way the conclusions of mathematics are consistent under our notions of consistency, but neither true nor false on account of the reasoning. And this is all that Russell is privileged to say when he asserts that "mathematics is the science in which we do not know whether the things we talk about exist nor whether our conclusions are true." From the results of logistic we certainly do not know either of these things. We merely know that if they exist, and if the premises are true, then the conclusions are true provided the processes of logistic can give true conclusions.

Since logistic does not touch the natural world, and since every one admits that mathematics does give us truth, the only possibility left to Russell was to assert the existence of a suprasensible world, the world of universals of Plato, in another form. In mathematics, he says, we are studying this world and making discoveries in it. It exists outside of the existence of any individual mind, and its laws are the laws of logistic naturally. That such world exists we will readily admit, yet we deny that it stands finished as a Greek temple in all

its cold and austere beauty, but that it is rather a living organism similar to the earth in geologic times, and out of the stress of temperature and moisture and dazzling sun there is evolved through the ages a succession of increasingly intricate and complex forms. But these forms derive their existence from the push and surge of the human mind beating against the cliffs of the unknown. Even logistic itself is the outburst of the mind from the barriers of the early attempts to think and to think clearly. Mathematics finally attacked even the process of thinking, just as it had considered number, space, operations, and hyper-number, and created for itself a more active logic. That this should happen was inevitable. Says Brunschvicg:⁸

"Symbolic logic, like poetic art following the spontaneous works of genius, simply celebrates the victory or records the defeat. Consequently it is upon the territory of positive science that the positive philosophy of mathematics should be placed. It gives up the chimerical ideal of founding mathematics upon the prolongation beyond the limits imposed by methodical verification itself of the apparatus of definitions, postulates, and demonstrations; it becomes immanent in science with the intention of discerning what is incorporated therein of intelligence and truth."

The philosophic assumption at the root of the view taken by the supporters of logistic as the sole source of truth we are not much concerned with, since we are not discussing philosophy but mathematics. But we may inspect it a little with profit. This assumption is the very old one, that there is an absolute truth independent of human existence and that by searching we may find it out. Says Jourdain:⁹

"At last, then, we arrive at seeing that the nature of

⁸ *Les étapes de la philosophie mathématique*, p. 426.

⁹ *Nature of Mathematics*, p. 88.

mathematics is independent of us personally and of the world outside, and we can feel that our own discoveries and views do not affect the truth itself, but only the extent to which we or others can see it. Some of us discover things in science, but we do not really create anything in science any more than Columbus created America. Common sense certainly leads us astray when we try to use it for purposes for which it is not particularly adapted, just as we may cut ourselves and not our beards if we try to shave with a carving knife; but it has the merit of finding no difficulty in agreeing with those philosophers who have succeeded in satisfying themselves of the truth and position of mathematics. Some philosophers have reached the startling conclusion that truth is made by men, and that mathematics is created by mathematicians, and that Columbus created America; but common sense, it is refreshing to think, is at any rate above being flattered by philosophical persuasion that it really occupies a place sometimes reserved for an even more Sacred Being."

Doubtless if Columbus were to discover America over again he might conclude that acts of creation had gone on in the meantime, and might reasonably assume that they happened in the past, and doubtless Mr. Jourdain is forced to conclude from his own argument that the words he uses in the English tongue have not been built up by the efforts of man but have existed from the beginnings of time, that the idea of propositional function and of relative and of function, pointset, transfinite number, Lobachevskian space, and a long list of other terms, have always been waiting in the mines of thought for the lucky prospector, but common sense would refute this view with very little study of the case. We may grant that electric waves have always existed, but that the wireless telegraph has always existed in any sense is not true; nor that even if carbon, nitrogen, hydrogen and oxygen have always existed, nitroglycerine

is to be dug out of wells, or that because sound-waves exist in the air, that therefore symphonies, operas, and all music have always been waiting to be discovered, not created. It is true perhaps that the elementary units out of which things material or mental are constructed exist in some sense, external to any one individual in some sense, but it is not true that therefore the combinations of these elements have always existed. Logistic, with all its boasted power, has never constructed a theorem that was truly synthetic in character, it has never taken a set of new postulates not derived from previously existing theories and developed a branch of mathematics similar to geometry or an algebra. It is powerless to move without the constant attendance of the intellect, it draws no more conclusions than Jevons's logical machine without its operator. It has never even introduced as one of its results a new thought of wide-reaching power, such as the idea of propositional function itself. This idea came from the extension of the mathematical function to other things than quantity. Columbus did not create the trees nor Indians nor shores of America, but he did create something that the Icelanders or the Chinese or other reputed previous discoverers did not create, and its existence we celebrate to-day more than the forgotten Indians, or the shifting sands of Watling's Island, or the broken tree-trunks. Mathematics, as we said before, did not spring like Athena from the head of Zeus, nor is it the record of the intellectual microscope and scalpel, but rather as Pringsheim,¹⁰ who is not a philosopher but a mathematician, says: "The true mathematician is always a good deal of an artist, an architect, yes, a poet. Beyond the real world, though perceptibly connected with it, mathematicians have created an ideal world which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world

¹⁰ *Jahr. Deut. Math. Ver.*, Vol. XXXII, p. 381.

except him who knows it; only presumptuous ignorance can assert that the mathematician moves in a narrow circle. The truth which he seeks is, to be sure, broadly considered, neither more nor less than consistency; but does not his mastership show indeed in this very limitation? To solve questions of this kind he passes unenviously over others."

We must pass on however to the reef that wrecked logistic in its short voyage after imperial dominion. This is nothing less than infinity itself. Since logistic asserted philosophically the suprasensible and supramental existence of its objects, it was forced to assert that there is an absolute infinity. In the transfinites of Cantor it found ultimately its ruin. In order to handle classes that had an infinite number of members it had to set up definitions that ultimately led to the contradictions which in the *Principles of Mathematics* of Russell were left unsolved. These were the objects of the assaults of Poincaré and others, and led to the definitive abandonment of the second volume of the *Principles*. The presentation of the *Principia* has many modifications, too long to cite, but the discussions in the *Revue de métaphysique et de morale* from 1900 on will be found very illuminating in their bearing on the nature of mathematics. The philosophical writings of Poincaré particularly should be consulted. The net result of all the discussions is that all the metaphysics has been eliminated from logistic, and it assumes its proper place in the mathematical family, as a branch of mathematics on a par with the other branches, such as arithmetic, geometry, algebra, group-theory.

The question of infinity is one of the most difficult to consider, and in one of his last articles Poincaré despairs of mathematicians ever agreeing upon it. The reason for perpetual disagreement he gives is the fundamental difference in point of view of reasoning in general. If the objects of mathematics are supramental, then the mind is forced

to admit an absolute infinity. If the objects of mathematics are created by the mind, then we must deny the absolute infinity. So far no decisive criterion has appeared, beyond that laid down by Poincaré, that any object about which we talk or reason must be defined, that is, made to be distinguishable from all other objects, in a finite number of words. For example, there is no such thing as the collection of all integers, since while we may define the class of integers and also any one integer we cannot define each and every integer. When logistic seeks to correlate the collection of all integers to any other infinite collection, member to member, this criterion demands that a law of correlation be stated which may be applied to every member of the class. This is manifestly impossible. A case is the proof that rational numbers may be put into a one-to-one correspondence with the integers. While any one rational may be placed in this way, or any finite number of them, yet according to the criterion it is not possible to decide that we can place every rational in this way. Manifestly any operation that has to be done in successive steps will never reach an absolute infinity. All proofs relating to infinite collections consider that the statement of a law for any member of the class is sufficient. The criterion demands a law for every member, which is admittedly not possible. The absolute infinity must not be confused with the mathematical infinity, which is merely an unlimited or arbitrary class. In all the processes we use in getting limits, the infinity that enters is not the Cantor infinity.

We may then safely conclude that logistic furnishes truth to the other branches of mathematics in much the same way that algebra does to geometry, or geometry to algebra, or numbers to group-theory, or hypernumbers to geometry. By logistic we may draw conclusions about the elements with which we deal. If we try to interpret the

conclusions logistic is powerless to do so any more than geometry can yield us theorems in logic. Also the processes in reasoning of any kind are no different in logistic from what they are in algebra, geometry, theory of numbers, theory of groups, and it is the intelligence, not the logistic, that draws the conclusion of logistic, just as it is the mathematician that solves algebraic equations, not algebra. Logistic has a right therefore to exist as an independent branch of mathematics, but it is not the Overlord of the mathematical world. As to the philosophical import of logistic, we may well follow Poincaré's advice, and continue the development of mathematics with little concern whether realism or idealism or positivism is substantiated in the philosophical world. Indeed we may conclude eventually with Lord Kelvin¹¹ that "mathematics is the only true metaphysics."

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URBANA, ILLINOIS.

¹¹ *Life*, p. 10.

RICHARD DEDEKIND.

(1833-1916.)

ON February 12, 1916, Julius Wilhelm Richard Dedekind died at his native Brunswick in Germany. He was one of the world's most distinguished workers at the theory of numbers, and in particular with Ernst Eduard Kummer and Leopold Kronecker at the theory of algebraic numbers; and most of his work is described in supplements to his editions of Dirichlet's *Vorlesungen über Zahlentheorie*.¹ In these supplements we can find references to his fundamental and enormously important ideas on the nature and meaning of numbers.

From the point of view of the fundamental principles of mathematics and the closely allied questions of logic and philosophy, the most important works of Dedekind are on the explanation of "continuity" by comparison with the system of real numbers, in which the irrational numbers were defined in a memorable way, and on the exceedingly subtle question of the definition, by logical concepts alone, of the integer numbers. Both of Dedekind's classical pamphlets: *Stetigkeit und irrationale Zahlen* of 1872 and *Was sind und was sollen die Zahlen?* of 1888 have been translated into English by W. W. Beman under the title: *Essays on the Theory of Numbers: I. Continuity and Irrational*

¹ A short indication of Dedekind's mathematical works was given by G. B. Mathews in *Nature*, Vol. XCVII, 1916, pp. 103-104.

*Numbers; II. The Nature and Meaning of Numbers.*² It is to this translation that the notes below refer.

The ideas of Dedekind on the nature and meaning of numbers, which are here described (§ II) after his logically subsequent and historically earlier work on continuity (§ I), led Dedekind to work out—apparently in complete independence of the previous work of De Morgan and the contemporary work of Charles Peirce—the greater part of what is now known as “the logic of relations.” On another occasion I hope to give an account of later critical and constructive work on both these contributions of Dedekind to the principles of mathematics.

I.

In the autumn of 1858, Dedekind, who was then professor at the Polytechnic School of Zurich, had, for the first time in his life, to lecture on the elements of the differential calculus, and then felt more acutely than ever before the lack of a really scientific foundation of arithmetic. “In discussing,” he said, “the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits must certainly approach a limiting value, I had recourse to geometrical evidences. Even now I maintain that such an employment of geometrical intuition is, from a didactic standpoint, extraordinarily useful and indeed indispensable, if we do not wish to lose too much time. But no one will deny that this manner of introduction to the differential calculus can make no claim to scientific accuracy. In my own case this feeling of dissatisfaction was so overpowering that I made a firm resolve to meditate until I should find a purely arithmetical and completely rigorous foundation for the principles of infinitesimal analysis.

² Chicago and London: The Open Court Publishing Co., 1901.

People say so often that the differential calculus is occupied with continuous magnitudes, and yet nowhere is there given an explanation of this continuity; and even the most rigorous expositions of the differential calculus do not found their proofs on continuity, but appeal with more or less consciousness of the fact to geometrical notions or notions suggested by geometry, or rest on theorems which have never been proved arithmetically. To these belongs, for example, the above mentioned theorem, and a closer investigation convinced me that this or any equivalent theorem can be regarded, in a sense, as a sufficient foundation for infinitesimal analysis. So all reduced to the discovery of its real origin in the elements of arithmetic and thus to obtain at the same time an actual definition of the essence of continuity. I succeeded in doing this on November 24, 1858." Although Dedekind communicated his ideas and discussed them with some of his colleagues and pupils, he could not make up his mind for many years to let them be printed because "the exposition is not quite easy, and besides the matter itself is so unfruitful."³ However, he had half determined to select that theme for a publication to be dedicated to his father on the celebration in April, 1872, of the fiftieth anniversary of his father's entry into office, when, in March of that year, he came across Heine's memoir in Vol. LXXIV of Crede's *Journal*, with which in essentials Dedekind agreed, "as indeed cannot be otherwise," but the form of his own work appeared to him to be simpler and to emphasize more precisely the main point. Also Dedekind remarked the identity of his axiom of the continuity of the straight line with Cantor's axiom, of which he read when writing his preface, and that he could not recognize the utility of Cantor's distinction of real numbers of still higher kind, because of his conception of the real number domain as complete in itself.

³ *Stetigkeit*, (2d ed., 1892), p. 2; cf. *Essays*, p. 2.

Comparing the system of rational numbers, in order of magnitude, with the points of a straight line L , we see that, if any origin be taken on L and a fixed unit of measurement, to any rational number a can be constructed a corresponding point; but there are points (those determined by incommensurable lengths measured from o) to which no rational numbers correspond. Thus we can say that " L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals."⁴ So if, as we wish,⁵ all phenomena in the straight line are also to be followed out arithmetically⁶ R must be refined by the creation of new numbers, and the domain of numbers raised to the same completeness—or "continuity"—as the straight line.

"The way in which irrational numbers are usually introduced is connected with the concept of extensive magnitude—which itself is nowhere rigorously defined—and explains number as the result of the measurement of one such magnitude by another of the same kind."⁷ Instead of this I demand that arithmetic shall be developed out of itself. That such connections with non-arithmetical notions have furnished the immediate occasion for the extension of the number-concept may, in general, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign connections into arithmetic, the science of numbers. Just as negative and fractional rational numbers must and can be formed by a new creation, and as the laws of operation with these numbers must and can be reduced to the laws of operation with posi-

⁴ *Stetigkeit*, p. 9; *Essays*, p. 9. ⁵ "Was doch der Wunsch ist," *ibid.*

⁶ Cf. *Stetigkeit*, pp. 5-6, 10; *Essays*, pp. 4, 10.

⁷ "The apparent advantage of this definition of number in point of generality vanishes the moment we think of complex numbers. In my view, the conception of the ratio to one another of two magnitudes of the same kind can be clearly developed only after the irrational numbers have been introduced."

tive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. There only remains the question as to how to do this.”⁸

Now the essence of this “continuity” of L was found by Dedekind⁹ after long meditation to be: If all the points of L fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which generates this division. This, as Dedekind emphasized, will probably be considered as evidently true by every one; it cannot be proved, but is an axiom by means of which we first recognize the line of its continuity. If space has a real existence, it need not necessarily be continuous; many of its properties would remain the same if it were discontinuous¹⁰; and if we knew that it was discontinuous, nothing could prevent us, if we wished, making it continuous in thought by filling up its lacunae. Another simple logical transformation of the above axiom is not so obvious: there is one and only one point (of the first class) which is on the extreme left of the first class, or one and only one of the second class on the extreme right of the second class, but not both.

⁸ *Steigkeit*, p. 10; *Essays*, pp. 9-10.

⁹ *Stetigkeit*, p. 11; *Essays*, p. 11. This axiom has been frequently misunderstood; thus L. Couturat (*De l'infini mathématique*, Paris, 1896, p. 416) stated it: “If all the quantities of a kind can be divided into two classes such that all the quantities of the one precede (or follow) all those of the other, there exists a quantity of this kind which *both* follows all those of the inferior class *and* precedes all those of the superior class.” Russell, in a review (*Mind*, Vol. VI, 1897, p. 117), rightly pointed out the mistake in this wording but wrongly advanced the same criticism against Dedekind’s own axiom (*The Principles of Mathematics*, Vol. I, Cambridge, 1903, p. 279). In fact, we do not need, as Russell presumed, a “point left over to represent the section”; and Russell’s (second) “emendation” (pp. 279-280) is Dedekind’s original axiom.

¹⁰ An example of this was given in the preface of *Was sind und was sollen die Zahlen?* (*Essays*, pp. 37-38). Choose any three points A, B, C, which do not lie in a straight line and which are such that the ratios of their distances AB, AC, BC are algebraic numbers; and regard as present in space only those points M for which the ratios of AM, BM, and CM to AB are algebraic numbers. The space consisting of the points M is everywhere discontinuous, but yet in it all the constructions in Euclid’s *Elements* can be carried out just as well as in a continuous space.

The purely arithmetical definition of new numbers among those of the system R so as to make it a continuous system was now brought about on a basis analogous to that of the above axiom. Any rational number a brings about a division of the system R into two classes A_1, A_2 , such that any number of A_1 is smaller than any number of A_2 ; a is either the greatest of A_1 or the least of A_2 . If now we have any division of R into classes A_1, A_2 , such that any member of A_1 is smaller than any member of A_2 , we call such a division a "section" or "cut" (*Schnitt, coupe*), and denote it by (A_1, A_2) . We can then say that any rational number a generates a section, or strictly speaking two sections (which, however, we will not regard as essentially different).¹¹ But there are an infinity of sections—such as that where A_1 consists of all the rational numbers r such that $r^2 < D$ is a positive non-square integer, and A_2 of the rest—which are not generated by rational numbers,—that is to say, neither has A_1 a maximum nor A_2 a minimum; and in this consists the incompleteness or discontinuity of R . Now, whenever we have a section (A_1, A_2) generated by no rational number, we create (*erschaffen*) a new, an "irrational number," which we regard as completely defined by the section (A_1, A_2) and is said to generate it.¹²

By comparing two sections, (A_1, A_2) and (B_1, B_2) , as to the inclusion or not of any term of A_1 in B_1 , or *vice versa*, we arrive at a basis for determining the order of any two real (rational or irrational) numbers α and β as symbolized by

$$\alpha = \beta, \alpha > \beta, \text{ or } \alpha < \beta,¹³$$

and also definitions of new sections whose generators may be represented by

¹¹ *Stetigkeit*, p. 12; cf. *Essays*, p. 13.

¹² *Stetigkeit*, p. 14; *Essays*, p. 15.

¹³ *Stetigkeit*, pp. 15-19; *Essays*, pp. 15-21.

$$\alpha + \beta, \alpha - \beta, \alpha \cdot \beta \text{ and } \alpha^\beta,$$

may be given.¹⁴

We will now indicate the use of the conception of a section to prove the theorems on limits mentioned above.¹⁵ A variable x is said to have a fixed limiting value α , when $x - \alpha$ ultimately sinks, numerically speaking, below any positive, non-zero, number; and our first theorem is that, if x increases continually, but not beyond all values, it approaches a *definite* limit. By the supposition, we have numbers α_2 such that we always have $x < \alpha_2$; denote the totality of these numbers by A_2 , and that of the other real numbers by A_1 . Any member (α_1) of A_1 has the property that in course of the process $x \leq \alpha_1$, and so every member of A_1 , is smaller than any member of A_2 , so that (A_1, A_2) is a section. Its generator (α) is either the greatest in A_1 or the least in A_2 ; the former cannot be the case, because x never ceases to increase. Thus α is the least member of A_2 , and it is a limit of the x 's, for, whatever member of A_1 the number α_1 may be, we ultimately have $\alpha_1 < x < \alpha$.

Still more often used is the equivalent of this theorem: If, in the process of variation of x , for any positive δ (however small) a corresponding place can be given from which one x varies by less than δ , then x approaches a limiting value. This can easily be derived from the foregoing theorem, or directly, as we do here, from the principle of continuity.

If $x = a$ at the instant referred to in the theorem, ever afterwards $x > a - \delta$ and $x < a + \delta$. On this fact we found a double separation of the system of real numbers. Put every number α_2 such that, in the course of the process, we have $x \leq \alpha_2$, in a class A_2 , and let A_1 consist of all the other numbers; so that, if α_1 is such a number it will happen

¹⁴ *Stetigkeit*, pp. 19-22; *Essays*, pp. 21-24.

¹⁵ *Stetigkeit*, pp. 22-24; *Essays*, pp. 24-27.

infinitely often, however far the process may have progressed, that $x > \alpha_1$. Since any α_1 is less than any α_2 , there is a definite generator α of the section (A_1, A_2) , which we will call the upper limiting value of x . Similarly, a second section (B_1, B_2) of the system of real numbers is brought about by x , if any number β_1 (such as $\alpha - \delta$) such that in the course of the process $x \leq \beta_1$ is put in B_1 ; and the generator β is called the lower limit of x . The two numbers α and β are also evidently characterized by the property that, if ϵ is taken positive and arbitrarily small, we always have $x < \alpha + \epsilon$ and $x > \beta - \epsilon$, but never finally $x < \alpha - \epsilon$ and never finally $x > \beta + \epsilon$. Now, two cases are possible: if α and β are different from one another (so that $\alpha > \beta$), x oscillates, and suffers, however far the process may have progressed, variations whose amount exceeds $(\alpha - \beta) - 2\epsilon$. But the original supposition, which is now first used, excludes this, and so there only remains the case $\alpha = \beta$; and we see that x approaches the limiting value α .

Dedekind remarked¹⁶ that, while the lengthiness in the definitions of the elementary operations can partly be overcome by the use of auxiliary concepts such as that of an "interval" (a system of rational numbers such that, if a and a' are any members of it, all the numbers between a and a' are also members of it)¹⁷ and of its limits, yet "still lengthier considerations seem to loom up when we wish to transfer the innumerable theorems of the arithmetic of rational numbers, as, for example, the theorem $(a + b)c = ac + bc$, to any real numbers. However, this is not so, for we soon convince ourselves that here all reduces to proving that the arithmetical operations themselves have a certain continuity. What I mean by this I will put in form of a general theorem: If the number λ is the result of a calcula-

¹⁶ *Stetigkeit*, pp. 20-22; *Essays*, pp. 22-24.

¹⁷ Both the classes of any section are "intervals."

tion undertaken with the numbers $\alpha, \beta, \gamma, \dots$, and if λ lies inside the interval L , then intervals A, B, C, \dots , inside which $\alpha, \beta, \gamma, \dots$, respectively lie, can be given such that the result of the same calculation in which $\alpha, \beta, \gamma, \dots$ are replaced by any numbers of A, B, C, \dots respectively, is always a number lying inside L . The forbidding clumsiness, however, which marks the enunciation of such a theorem convinces us that here something must be done to aid language. This is done in the most satisfactory way by introducing the concepts of *variable magnitudes*, *functions*, and *limiting values*; and indeed the most convenient thing is to base the definitions of the simplest arithmetical operations on these concepts, but this cannot be carried farther here.”¹⁸

II.

The last few words contain an indication of the fundamental concepts upon which Dedekind’s theory of integers was based. The notion of an aggregate or “system” of things is, of course, the most fundamental, and also we utilize, in counting, the capability of the mind to *refer* things to things, to let a thing *correspond* to a thing, or to *image* (*abzubilden*) a thing by a thing. Without this capability no thought is possible, and on this single, but quite indispensable, foundation must, in Dedekind’s view, the whole science of numbers be erected.¹⁹ This idea of

¹⁸ *Stetigkeit*, pp. 21-22; cf. *Essays*, pp. 23-24.

¹⁹ In the eleventh appendix of Dedekind’s edition of Dirichlet’s *Vorlesungen über Zahlentheorie* (3d ed., 1879, § 163, p. 470), Dedekind said: “It happens very frequently in mathematics and other sciences that, if we have a system Ω of things or elements ω , every definite element ω is replaced according to a certain law by a definite element ω' corresponding to it. We are accustomed to call such an act a substitution and say that by this substitution the element ω passes over into the element ω' and the system Ω into a system Ω' of elements ω' . The expression of this is somewhat more convenient if we ... conceive this substitution as a transformation (*Abbildung*) of the system Ω .” To this he added the note: “On this ability (*Fähigkeit*) of mind to compare a thing ω with a thing ω' , or to refer ω to ω' , or to let ω correspond to ω' , without which thought is impossible, rests, as I will try to prove in another place, the whole science of numbers.”

correspondence is the idea of functionality or, in other words, of establishing a *relation* between things.

Dedekind's views on the nature of numbers may be expressed as follows. Arithmetic, including Algebra and Analysis, "is a part of logic, and the number-concept is quite independent of the notions or intuitions of space and time, and is an immediate consequence of the pure laws of thought." Toward the beginning of his *Stetigkeit*,²⁰ he wrote: "I regard the whole of arithmetic as a necessary or at least natural consequence of the simplest arithmetical act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers, in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already formed individual to the consecutive new one to be formed. The chain of these numbers forms even by itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any repetitions we wish of the above mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been and whatever comparisons or analogies with experience or intuition may have led us, it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act. Thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater perfection. Numbers are free creations of the human mind; they serve as a

²⁰ Pp. 5-6; cf. *Essays*, p. 4.

means to grasp the difference of things more easily and distinctly. Only by means of the purely logical structure of the science of numbers and the continuous number-region obtained in it are we in a position accurately to investigate our notions of space and time, by referring them to this number-domain created in our mind."

Dedekind had the intention of showing the development of the conception of the natural (integral) numbers from the purely logical conceptions of aggregate and "representation" (*Abbildung*), before the publication (1872) of his work on continuity, but it was only after the appearance of this work that, from 1872 to 1878, he wrote out a sketch of his system containing all its essential ideas, and showed it to and discussed it with many mathematicians. In 1887 a careful exposition was carried out and published in the next year under the title *Was sind und was sollen die Zahlen?*²¹ The motive for the publication was the appearance of the essays of Kronecker and von Helmholtz. His own work, as he said, though similar in many respects to those essays, was in its foundations essentially different, and he had formed his own view "many years before and without influence from any side."

Dedekind regarded the maxim that "in science anything which can be proved is not to be accepted without proof"²² as unfulfilled even in the most recent methods of laying the foundations of arithmetic. And Dedekind's answer to this want was one of the first examples of that tendency of modern mathematics to extend exactness of treatment to the very principles, that has been gradually carried out by mathematical logicians like Frege, Peano and Russell.

As we should expect, the tract at first excited the derision of those unperceiving mathematicians who thought

²¹ Brunswick, 1888; second unaltered edition, 1893 [prefaces dated Oct. 5, 1887 and Aug. 24, 1893]; *Essays*, pp. 31-115.

²² *Was sind und was sollen die Zahlen?*, p. vii; cf. *Essays*, p. 31.

that Dedekind was merely taking an unnecessarily long time to prove obvious things like the commutative law in arithmetic. That such things seem to be immediately obvious will at once be granted, but the logical problem which interested Dedekind and many others since about the middle of the nineteenth century was whether or not such theorems are logically implied by those (logical) principles which hold for all true thought without exception, and are not of merely empirical validity. If we are in sympathy with efforts to solve the problems of the nature of our knowledge, we ought not to complain that the detailed writing out of logical steps takes up a large space. Besides, such a complaint is irrelevant.

Dedekind considered what he called "systems," which are what logicians call "classes" and mathematicians now usually call "aggregates," and then the idea of a correspondence of the elements of a system with elements of another system or with one another. He viewed such a correspondence as a "transformation"; and, when he came to consider "similar [or one-to-one] transformations of a system into a part of itself," he arrived at defining an "infinite" system²³ and thus fell upon much the same ideas that Georg Cantor independently did.²⁴ A special infinite system is the "simply infinite system" N which is such that there exists a similar transformation φ such that $\varphi(N)$ is a part of N , and N is the common part of all systems S which contain a definite element of N which is not of $\varphi(N)$, and for which $\varphi(S)$ is a part of S .²⁵ We can see without much difficulty that N consists of an element a , its transform a' , the transform a'' of a' , and so on; but it is to be noticed that Dedekind defines his infinite systems as wholes and does not use the vague words "and so on."

²³ *Essays*, pp. 63, 41-42.

²⁴ *Contributions to the Founding of the Theory of Transfinite Numbers*, Chicago and London, 1915, p. 41.

²⁵ Cf. *Essays*, pp. 67, 56-58.

The ordinal numbers then appear as mental abstractions from such systems as N ,²⁶ the theorem of complete induction is proved for them,²⁷ and the various other fundamental arithmetical concepts and theorems established. In particular, Dedekind considered cardinal numbers to be logically subsequent to ordinal numbers.²⁸

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²⁶ *Ibid.*, p. 68.

²⁷ *Ibid.*, pp. 69-70, 60-62, 32-33, 42-43.

²⁸ *Ibid.*, pp. 109-110, 32.

CRITICISMS AND DISCUSSIONS.

THE ARITHMETICAL PYRAMID OF MANY DIMENSIONS

AN EXTENSION OF PASCAL'S ARITHMETICAL TRIANGLE TO
THREE AND MORE DIMENSIONS, AND ITS APPLI-
CATION TO COMBINATIONS OF MANY
VARIATIONS.

I.

In 1665 Pascal wrote his *Traité du triangle arithmétique* and showed that the system of numbers there developed, the so-called *figurate* numbers, had many remarkable properties. The most useful of these, and for our present purposes the most important, is the fact that this table gives the value of the expression ${}_nC_r$, for all positive integral values of n and r (including 0). The expression ${}_nC_r$ means the number of combinations of n things taken r at a time.

It is also written $\binom{n}{r}$, and is equal to

$$\frac{n(n-1)(n-2)(n-3)\dots(n-r+1)}{r!}$$

or $n!/r!(n-r)!$, in which $r!$ is read "factorial r " and denotes the product of all the integral numbers from 1 to r inclusive. The appropriate solution for any given values of n and r is to be found in the n th line and r th column of the arithmetical triangle. See Table I.

Now ${}_nC_r$ refers to things, each of which is capable of two and only two variations, such as coins that may fall either heads or tails. But frequently we have to do with things subject to more than two variations, such as a number of signal lights each showing several colors, or a number of dice which may fall on any one of their six faces. The solutions of such cases are not to be found in the arithmetical triangle, though in every case they can be shown to be

Names of the Columns	Natural Nos.	Triangular Nos.	Tetrahedral Nos.	Pentahe-droidal Nos.	Hexahedroidal Nos.	Octahedroidal Nos.	Nonahedroidal Nos.	Dekahedroidal Nos.	Undekahedroidal Nos.	
Boundaries of the Figures	Corners	Edges	Sur-faces	Tetra-he-dra	Penta-he-dra	Hexa-he-dra	Septa-he-dra	Octa-he-dra	Nona-he-dra	Deka-he-dra
	0	1	2	3	4	5	6	7	8	9
	0	1								1
Point	1	1	1							2
Line	2	1	2	1						4
Triangle	3	1	3	3	1					8
Tetrahedron	4	1	4	6	4	1				16
Pentahedroid	5	1	5	10	10	5	1			32
Hexahedroid	6	1	6	15	20	15	6	1		64
Septahedroid	7	1	7	21	35	35	21	7	1	128
Octahedroid	8	1	8	28	56	70	56	28	8	256
Nonahedroid	9	1	9	36	84	126	126	84	36	512
Dekahedroid	10	1	10	45	120	210	252	210	120	45
Combinations	nC_0	nC_1	nC_2	nC_3	nC_4		nC_{n-2}	nC_{n-1}	nC_n	
General Formulas	$0! \cdot (n-0)!$	$1 \cdot \frac{n!}{(n-1)!}$	$2 \cdot \frac{n!}{(n-2)!}$	$3 \cdot \frac{n!}{(n-3)!}$		$\frac{n!}{(n-4)!}$	$\frac{n!}{(n-5)!}$	$\frac{n!}{(n-6)!}$	$\frac{n!}{(n-7)!}$	2^n

TABLE I. The Arithmetical Triangle.
See also Table VI for another arrangement.

the product of two or more numbers there to be found. So far as the writer is aware no systematic method of selecting the proper factors has yet been given.

In the case of two variations, for any given value of n there will be $n+1$ classes, obtained by giving r successively all integral values from 0 to n . In any class r is the number of one kind present, $n-r$ the number of the other. These can all appropriately be arranged along a straight line. In fact the complete set of solutions is to be found in the n th line of the arithmetical triangle. But if the n things are capable of more than two variations—if for example they may be A's, B's, C's, D's, etc.—then a much larger number of classes arises; for to any one of these letters may be assigned in turn all the integers from 0 to n , and all vary independently. These classes cannot be so simply arranged, and the task of obtaining all of them and calculating the number of combinations for each becomes very complicated. Some systematic method must be adopted to insure exhaustive enumeration.

The object of the present paper is to show how these cases of many variations may be appropriately arranged in more-dimensional tables, so as to develop with certainty all possible classes, and show their proper relations to one another, and also to show how the arithmetical triangle may likewise be extended to more dimensions, and thus provide means of readily finding the number of combinations corresponding to each class. The method is somewhat complicated to explain, but easy to operate. We shall begin by describing a few of the many remarkable properties of the arithmetical triangle, such as will be useful to us, and then take up in turn its extension to 2, 3, 4, . . . , k variations.

All the numbers of the arithmetical triangle can of course be calculated from the general formula already given, $n!/r!(n-r)!$. But the table can also be much more simply produced by a process of successive addition as follows: Beginning with 1, below any line write the same line moved one place to the right and add. The result is the next line. The process is shown in Table II.

From the mode of development it is apparent that the differences of any column are to be found in the next column to the left. Any column is therefore an arithmetical series of the r th order, whose r th differences are constant and equal to 1. The table is in fact the complete system of all arithmetical series whose final differences are 1. Conversely each number gives the sum of all the

preceding numbers of the next column to the left, or the sum of any two numbers in the same line is found immediately below the right-hand one.

Each line gives the binomial coefficients in order for the exponent corresponding to the number of the line, for these coefficients are also given by the formula $\binom{n}{r}$. The sum of all the numbers of any line is 2^n .

The columns have been given special names because of certain properties they possess. The zero column is composed only of units. The first column contains the natural numbers. The second contains the triangular numbers, so called because they give the number of units that can be arranged in a triangle, having suc-

Line Zero	1
	1
Line One	<u>1 1</u>
	1 1
Line Two	<u>1 2 1</u>
	1 2 1
Line Three	<u>1 3 3 1</u>
	1 3 3 1
Line Four	<u>1 4 6 4 1</u>

TABLE II. Method of Constructing Arithmetical Triangle.

sively 1, 2, 3, 4, . . . units on a side. The third column contains the pyramidal numbers, so called because they give the number of units that can be piled like cannon balls in the form of a triangular pyramid or tetrahedron, having successively 1, 2, 3, 4, . . . units on a side.

The remaining columns have as yet received no special names, but they might appropriately be named after the succeeding higher-dimensional pyramids, since they similarly give in turn the numbers of units that can be arranged in the form of these higher pyramids, having successively 1, 2, 3, 4, . . . units on a side. We shall call the latter, after Stringham, successively, the 4-dimensional pentahedroid, the 5-dimensional hexahedroid, the 6-dimensional septahedroid, etc., in general the $(k - 1)$ -dimensional k -hedroid, and name the columns

after them as shown in Table I. Thus the tetrahedron becomes a 4-hedroid, the triangle a 3-hedroid, the line a 2-hedroid, the point a 1-hedroid, and the corresponding columns the 4, 3, 2, 1-hedroidal numbers respectively.

Most useful for our subsequent purposes however is the fact that the arithmetical triangle gives complete specifications for the construction of any of these higher pyramids. Thus the n th line gives in order the number of 0, 1, 2, 3, ..., $(n-1)$ -dimensional boundaries of the $(n-1)$ -dimensional n -hedroid. We have only to read the line designating the succeeding numbers in turn as so many corners, edges, surfaces, tetrahedra, etc., as indicated in Table I. Thus we may read:

First line: One point has 1 corner or 0-space boundary.

Second line: One line has 2 corners or ends, and 1 edge or interior 1-space.

Third line: One triangle has 3 corners, 3 edges, and 1 interior 2-space or surface.

Fourth line: One tetrahedron has 4 corners, 6 edges, 4 surfaces, and 1 interior 3-space.

Fifth line: One pentahedroid has 5 corners, 10 edges, 10 triangular surfaces, and 1 interior space of four dimensions.

Similarly the remaining lines may be read.

It may be noted that as the line lies between its ends, the triangle within its edges, the 3-space of the tetrahedron inside its bounding 2-space surfaces, so the 4-space of the pentahedroid is *inside* its bounding 3-space tetrahedra. Similarly with the higher pyramids. The 5-space is inside the 4-space, the 6- inside the 5-, etc. We get to higher and higher degrees of insideness.

As we shall use these higher pyramids to represent our combinations of many variations, it is important to know how they are constructed.

We may now proceed to our task of applying the arithmetical triangle to the cases of two and more variations. Calling k the number of variations, it will be found in every case that a $(k-1)$ -dimensional k -hedroidal table will be required, the total number of classes is given by the $(n+1)$ th k -hedroidal number, while the sum of all the combinations is k^n . This gives a valuable check on the correctness of the work. The variations we shall call A, B, C, D, etc.

Let $k = 2$. The complete set of solutions for this case, as has already been stated, is to be found in the n th line of the arithmetical

triangle. They form therefore a linear table, as shown for $n=10$, in Table III. The second half of any such line is always the reverse of the first half, so that there are only $(n+2)/2$, disregarding remainder, different numbers in it.

	0	1	2	3	4	5	6	7	8	9	10	B
A	10	9	8	7	6	5	4	3	2	1	0	
	1	10	45	120	210	252	210	120	45	10	1	

TABLE III. All Classes Combinations of 10 Things each susceptible of 2 Variations, such as 10 Coins, that may fall Heads or Tails.

Number of classes = $n+1=11$. Sum of combinations = $2^{10}=1024$.

Note. First line shows number of B's present, second line number of A's.

Let $k=3$. Each thing may now be an A or a B or a C. The additional classes possible may be derived from the previous case in the following way. To each class of A and B, as shown in Table III, additional classes may be formed, by keeping the B's constant, and exchanging the A's one by one for C's. The number of new classes thus formed will in each case be equal to the number of A's contained in the original cell. Thus the first cell gives 10, the second 9, the third 8, etc. new classes. The only appropriate way to arrange the new classes is in columns vertically under the cells from which they were developed, according to descending values of A, or ascending values of C. In this way the triangular Table IV is obtained, for the value of $n=10$. It may be noted that there will be three classes where the n things are all of one kind, as all A's, all B's, or all C's. These are placed at the corners, or 0-dimensional boundaries of the triangle. There will be three sets of classes where the things are AB, BC, or AC. Each of these three sets is a duplicate of Table III, and they are located on the three edges of the triangle. The remaining classes in which all three variations appear are all located in the interior of the triangle, or in 2-space.

Further the triangle may be divided, as shown by the dash lines of Table IV, into a series of similar concentric triangular shells. The first interior shell contains all classes where one of the letters or variations appears each time only once. The second interior shell contains all classes where one of the letters appears always twice, the third three, etc. There will always be $n/3$, disregarding remainder, of these interior shells. It may further be noted that in each line,

column and diagonal, the second half is the reverse of the first half. The same numbers are hence oft repeated. In fact in Table IV there are only 14 different numbers, while if we deduct the 6 taken direct from Table III there are only 8 new numbers to calculate. These are shown enclosed in heavy lines in the table. It is easily seen that they cover approximately $\frac{1}{6}$ of the total area of the triangle, hence may be calculated for any n from $n^2/12$, taking the

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

Note. The first line shows number of B's present, left-hand column the number of C's. The remaining $n - (B + C)$ are A's.

Number of classes = 11th triangular number or $[(n+1)(n+2)]/2 = 66$.

Sum of all combinations = $3^{10} = 59,049$.

TABLE IV. Combinations of 3 Variations for $n = 10$.

nearest whole number. For the values of n from 1 to 15 the following are obtained as the new interior numbers to calculate:

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.$$

$$\text{Nos. } 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19.$$

The fact that for the first two cases no new numbers are obtained means that all the cells abut on the edges. When $n = 3$ we get 1 interior cell; when $n = 4$ we get 3 cells but all are filled with the same number, viz. 12, as the reader may verify, if he likes, by working out these simple cases. Hence the number of calculations to make is not very great unless n is large.

It will be noted of course that the numbers increase in size toward the geometric center of the triangle. Also from the method

of numbering, the B's are constant in any column, the C's along any line, and each increases in value from the right angle outward. The A's are constant along any diagonal and increase in value toward the right angle, where they are all A's. This method of lettering and numbering will be adhered to in all that follows, and the characteristics that depend upon it will naturally always recur.

The value of any interior number is given by the general formula

$$\frac{n!}{A!B!C!} = \frac{n!}{B!C![n-(B+C)]!}, \text{ since } A+B+C=n. \text{ Calling}$$

$0!=1$, as is customary, this formula will also apply to the edges and hence to all the numbers of the triangle. But there is a much easier way of deriving the appropriate numbers directly from the arithmetical triangle, which it is one of the objects of this paper to show. If in the above formula we give to A, B and C different values, and write them down in their proper places, we obtain Table V, below. Since, as already stated, the A's, B's and C's are constant respectively along the diagonals, columns and lines it is apparent that the expression $n!/A!B!C!$ can, in three ways, be divided into two factors, one of which is constant and the other variable, according as we choose the constant part along a diagonal, a column or a line. Thus,—

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!} \times \frac{1}{C![n-(B+C)]!} \quad (1)$$

$$\text{or } = \frac{n!}{C!} \times \frac{1}{B![n-(B+C)]!} \quad (2)$$

$$\text{or } = \frac{n!}{[n-(B+C)]!} \times \frac{1}{B!C!} \quad (3)$$

In every case the first factor is constant, the second variable. In (1) first factor is constant along a column, in (2) along a line, in (3) along a diagonal. Not much is gained by this, but if we multiply numerator and denominator in the three cases respectively by $(n-B)!$, $(n-C)!$, $(B+C)!$, we obtain,

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!(n-B)!} \times \frac{(n-B)!}{C![n-(B+C)]!} = nC_B \times {}_{n-B}C_C \quad (4)$$

$$\text{or } = \frac{n!}{C!(n-C)!} \times \frac{(n-C)!}{B![n-(B+C)]!} = nC_C \times {}_{n-C}C_B \quad (5)$$

$$\text{or } = \frac{n!}{(B+C)![n-(B+C)]!} \times \frac{(B+C)!}{B!C!} = {}_{n-B+C}C_B \times {}_{n-C}C_C \quad (6)$$

In all three cases now each factor is seen to be a figurate number, hence one to be found in the arithmetical triangle. Moreover

A	0	1	2	3	4	5	6	B
0	$\frac{n!}{0!0! [n-(0+0)]!}$	$\frac{n!}{1!0! [n-(1+0)]!}$	$\frac{n!}{2!0! [n-(2+0)]!}$	$\frac{n!}{3!0! [n-(3+0)]!}$	$\frac{n!}{4!0! [n-(4+0)]!}$	$\frac{n!}{5!0! [n-(5+0)]!}$	$\frac{n!}{6!0! [n-(6+0)]!}$	
1	$\frac{n!}{0!1! [n-(0+1)]!}$	$\frac{n!}{1!1! [n-(1+1)]!}$	$\frac{n!}{2!1! [n-(2+1)]!}$	$\frac{n!}{3!1! [n-(3+1)]!}$	$\frac{n!}{4!1! [n-(4+1)]!}$	$\frac{n!}{5!1! [n-(5+1)]!}$		
2	$\frac{n!}{0!2! [n-(0+2)]!}$	$\frac{n!}{1!2! [n-(1+2)]!}$	$\frac{n!}{2!2! [n-(2+2)]!}$	$\frac{n!}{3!2! [n-(3+2)]!}$	$\frac{n!}{4!2! [n-(4+2)]!}$			
3	$\frac{n!}{0!3! [n-(0+3)]!}$	$\frac{n!}{1!3! [n-(1+3)]!}$	$\frac{n!}{2!3! [n-(2+3)]!}$	$\frac{n!}{3!3! [n-(3+3)]!}$				
4	$\frac{n!}{0!4! [n-(0+4)]!}$	$\frac{n!}{1!4! [n-(1+4)]!}$	$\frac{n!}{2!4! [n-(2+4)]!}$					
5	$\frac{n!}{0!5! [n-(0+5)]!}$	$\frac{n!}{1!5! [n-(1+5)]!}$						
6	$\frac{n!}{0!6! [n-(0+6)]!}$							
C								

TABLE V.
Formulas for Calculating Combinations of 3 Variations.

This table is of indefinite Extent, but for any given value of n comes to an end when $n = B + C$.

First factorial of denominator gives number of B's.
Second " " " " " " C's.
Third " " " " " " A's.

General formula: $\frac{n!}{B! C! [n-(B+C)]!}$

the first factor is not only constant but in (4) it is *equal* to the number that stands at the head (and foot) of the corresponding column, in (5) to the number that stands at either end of the corresponding line, in (6) to the number that stands at either end of the corresponding diagonal of the table to be calculated, while in each case the second factor for these terms becomes 1. By the first scheme then all the members of any column could be obtained by multiplying the first one by the successive values of the second factor of (4), obtained by giving B and C the proper values. Similarly by the second scheme all the terms of any line could be obtained from the first one, while by the third scheme all the terms of any diagonal from the end ones. Any one of the three schemes would produce the whole triangle, and it would seem natural to choose either of the first two. However because of the occurrence of $n - B$, and $n - C$, in these two schemes, making it necessary to assign a definite value to n before anything can be done, they do not lend themselves so readily to general treatment as the third scheme. We shall accordingly adopt the latter.

The proper values of the second factor, or coefficient as we shall call it, could of course be found in line B, or C, of column B+C of the arithmetical triangle. But if we give different values to B and C and write the resulting numbers down in their proper places in the triangle, we obtain Table VI. It is at once seen that they follow a very regular order, being in fact nothing other than the arithmetical triangle itself, with each column pushed up to the top. This is the usual arrangement of the figurate numbers. The diagonals of the new table are the lines of the old.

The procedure of calculating a triangle then is as follows. As the first line write the n th line of the arithmetical triangle. Any interior number is then calculated by multiplying the number that stands at the end of its diagonal by the coefficient shown in Table VI. The process is shown in Table VII. Or the process may be described thus: Each successive line is derived from the first line by discarding each time one additional term and multiplying the remaining terms first by the natural, then by the triangular, then by the pyramidal, etc., numbers in order. Or we may sum the whole thing up in one general rule:

The m th line of a "surface triangle" is derived from the n th line of the arithmetical triangle by discarding $m-1$ terms and multi-

plying the remaining terms by the *m*-hedroidal numbers in order, beginning with the first one.

This rule applies to the outside edge as well as to the inner lines. The reason for calling the figure a *surface triangle* is because the surfaces of all our subsequent higher pyramids will be composed of such figures.

We have used for purposes of illustration $n=10$. If we give to n other values we shall obtain similar triangles, smaller or larger according to the value of n . If we construct a series of these from $n=0$ up, and pile them all up on top of each other with their A

A	0	1	2	3	4	5	6	7	8	9	10	11	12	B
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11	12		Natural Nos.
2	1	3	6	10	15	21	28	36	45	55	66			Triangular Nos.
3	1	4	10	20	35	56	84	120	165	220				Pyramidal Nos.
4	1	5	15	35	70	126	210	330	495					Pentahedroidal Nos.
5	1	6	21	56	126	252	462	792						Hexahedroidal Nos.
6	1	7	28	84	210	462	924							Etc.
7	1	8	36	120	330	792								
8	1	9	45	165	496									
9	1	10	55	220										
10	1	11	66											
11	1	12												
C														

TABLE VI.

Coefficients for Calculating Combinations of 3 Variations or General Table of Figurate Numbers.

vertices coinciding, we shall obtain the *arithmetical pyramid* of three dimensions, as shown in Fig. 1. This has the form of a number of cubes piled in a corner of the room. It can of course be indefinitely extended. In it will be found all classes of combinations for $k=3$, just as in the plane arithmetical triangle are to be found all classes for $k=2$. Each surface will be an arithmetical of the old sort, all the new classes of three variations being found in the interior cells.

As the numbers ${}^n C_r$ are binomial coefficients in the expansion of $(a+b)^n$, so the numbers representing combinations of three

variations are the trinomial coefficients arising in the expansion of $(a+b+c)^n$. Thus Table IV enables us at once to write out the expansion of $(a+b+c)^{10}$. In like manner the numbers representing combinations of k variations are k -nomial coefficients arising in the expansion of the n th power of a polynomial of k terms.

Let $k=4$. Each thing may now be an A or a B or a C or a D. In order to develop and represent all possible classes in their proper relations to each other, use will now have to be made of a three-

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1 ×1	10 ×1	45 ×1	120 ×1	210 ×1	252 ×1	210 ×1	120 ×1	45 ×1	10 ×1	1 ×1	
1	10 ×1	45 ×2	120 ×3	210 ×4	252 ×5	210 ×6	120 ×7	45 ×8	10 ×9	1 ×10		
2	45 ×1	120 ×3	210 ×6	252 ×10	210 ×15	120 ×21	45 ×28	10 ×36	1 ×45			
3	120 ×1	210 ×4	252 ×10	210 ×20	120 ×35	45 ×56	10 ×84	1 ×120				
4	210 ×1	252 ×5	210 ×15	120 ×35	45 ×70	10 ×126	1 ×210					
5	252 ×1	210 ×6	120 ×21	45 ×56	10 ×126	1 ×252						
6	210 ×1	120 ×7	45 ×28	10 ×84	1 ×210							
7	120 ×1	45 ×8	10 ×36	1 ×120								
8	45 ×1	10 ×9	1 ×45									
9	10 ×1	1 ×10										
10	1 ×1											
C												

TABLE VII.

Method of Calculating Combinations for $n=10$, $k=3$, from 10th Line of Arithmetical Triangle, and Coefficients of Table VI. Result is Table IV.

Only the 8 numbers enclosed in heavy lines need be calculated, all the others being taken direct or repetitions.

dimensional arrangement. For to every class containing A, B and C, as shown in Table IV, additional classes may now be formed by exchanging the A's successively for D's. The only proper place to put these new classes is to build them out in the third dimension from the original cells of Table IV from which they were developed. The resulting arrangement will be pyramidal in form because of the regularly decreasing value of A from the corner outward to the last diagonal. Such a pyramid, as we read in the fourth line of the arithmetical triangle, will be composed of 4 corners, 6 edges,

4 surfaces and 1 interior space. These will carry respectively the classes where 1, 2, 3 and 4 variations are present. There will be the following groups of such classes:

1 Variation,	A, B, C, D,	$= {}_4C_1, = 4$
2 Variations,	$\begin{Bmatrix} AB & BC & CD \\ AC & BD \\ AD \end{Bmatrix}$	$= {}_4C_2, = 6$
3 Variations,	ABC ABD ACD BCD	$= {}_4C_3, = 4$
4 Variations,	ABCD	$= {}_4C_4, = 1$

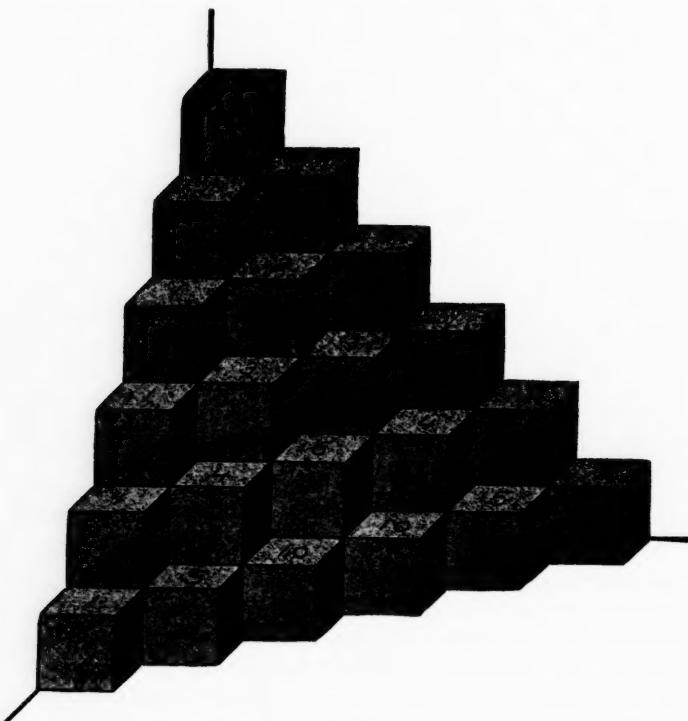


Fig. 1.

Each of the four surfaces of the pyramid will, for $n=10$, be exactly the same as the triangle of Table IV, only as the 12 edges

of the 4 triangles coincide in pairs, thus reducing to 6, and as the 12 corners coincide in threes, thus reducing to 4, we cannot simply repeat the whole triangle four times, but regard must be had for the corners and edges to be omitted. One way would be to represent the corners, edges and interior part of the triangles separately. Another method is shown in Fig. 2. Here the slant sides are supposed to be folded down into the plane of the base, so as to depict all in one plane. The triangle BCD, which is in reality equilateral, is conveniently made right-angled like the others. The sides to be omitted are indicated by dotted lines. A modification of this plan, that economises space, is shown in Fig. 3. A modification of the first method is used in Diag. 1, Table VIII, and in most of the

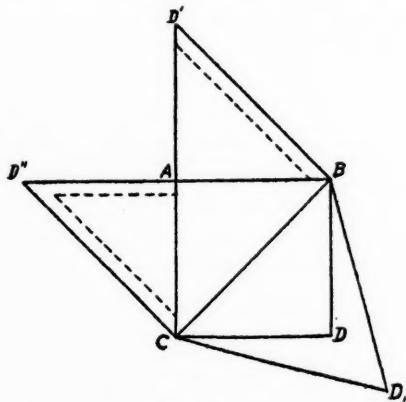


Fig. 2.

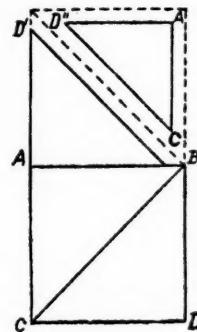


Fig. 3.

higher pyramids. The method of Fig. 2 is used in Diag. 5, Table XII, that of Fig. 3 in Diags. 2 and 3 of Table VIII.

The surface of the pyramid being thus represented, and no new numbers to calculate for it, it remains to consider only the representation and calculation of the interior cells, where four variations are present. To do this it will be found most convenient to consider the pyramid as made up of a number of concentric pyramidal shells, each one cell in thickness, like, to use an unsavory simile, the coatings of an onion. Each interior shell will then be exactly similar to the surface shell, which has just been peeled off. It can be represented in the same way, by any one of the three methods described. Analogous to the triangular shells of Table IV,

each cell of the first inner pyramidal shell will contain, beside the numbers indicated of the letters that stand at the vertices of the particular triangle in which it is found, one example of the missing letter. Thus the four triangles of the first inner shell are appropriately lettered, ABC + 1D, ABD + 1C, ACD + 1B, DCB + 1A, as in

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

TABLE VIII, DIAG. 1.
Typical Surface Triangle of Pyramid for
 $n = 10, k = 3$.

The pyramid has 4 corners, 6 edges, 4 surfaces.

Each corner has $1 \times 4 = 4$ or k classes.

“ edge “ $9 \times 6 = 54$ “ $6(n-1)$ “

“ surface “ $36 \times 4 = 144$ “ $4 \frac{(n-1)n}{2}$ “ ; or $(n-1)$ th Triangular No.

Total number of classes = 202.

Each corner has $1 \times 4 = 4$ or k Combinations

Each edge “ $1,022 \times 6 = 6,132$ or $6(2^n-2)$ “

Each inner surface “ $55,980 \times 4 = 223,920$ or $4[3^n-3(2^n-2)-3]$ “

Total comb. of surface shell = 230,056.

The four triangles are to be lettered, ABC, ABD, ACD, DCB, as in Figs. 2 and 3.

Diag. 2, Table VIII. Similarly the cells of the second inner shell contain two of the missing letter, and are lettered as in Diag. 3, Table VIII. The third shell, when there is one, will contain 3, the fourth 4, etc., of the missing letter. Each inner shell will have four less cells on a side than the next outer one. Hence until $n = 4$

D			D	5	4	3	2	A
6	2520			7560	12600	12600	7560	2
5	5040	7560			12600	16800	12600	3
4	6300	12600	12600			12600	12600	4
3	5040	12600	16800	12600			7560	5
2	2520	7560	12600	12600	7560			C
A	1	2	3	4	5	6	7	B
1	720	2520	5040	6300	5040	2520	720	7
2	2520	7560	12600	12600	7560	2520	2520	6
3	5040	12600	16800	12600	5040	7560	5040	5
4	6300	12600	12600	6300	12600	12600	6300	4
5	5040	7560	5040	12600	16800	12600	5040	3
6	2520	2520	7560	12600	12600	7560	2520	2
7	720	2520	5040	6300	5040	2520	720	1
C	7	6	5	4	3	2	1	D

TABLE VIII, DIAG. 2.

First Inner Shell of Pyramid.

No. of classes = 74. No. of comb. = 591,720. 7 Nos. to calculate.

D		D		A
3	25200			C
A	2	3	4	B
2	18900	25200	18900	4
3	25200	25200	25200	3
4	18900	25200	18900	2
C	4	3	2	D

TABLE VIII, DIAG. 3.
Second Inner Shell of Pyramid.

No. of classes = 10. No. of comb. = 226,800. 2 Nos. to calculate.

Summary of Pyramid.

4 Classes of 1 Variation having	4 Combinations.
54 " " 2 Variations "	6,132 "
144 " " 3 " "	223,920 "
84 " " 4 " "	818,520 "
286 " " all " "	1,048,576 "
	= 11th Pyramidal Number.
	= 4^{10} .

there will be no inner shells. The number of inner shells will always be $n/4$ disregarding the remainder. The first inner shell will be numbered from 1 to $n-3$, the second from 2 to $n-2 \cdot 3$, the third from 3 to $n-3 \cdot 3$; in general, if t is the number of the shell it will be numbered from t to $n-3t$. These inner shells we shall call for short the first tetra, second tetra, etc. They are not really tetrahedral in shape, being composed mostly of right-angled, instead of equilateral triangles, but it will be convenient to call them as designated.

In Table VIII is worked out this case for $n=10$. Diag. 1, which represents the typical surface triangle ABC, is the same as in Table IV. Only the interior portion, enclosed in heavy lines, is exactly repeated on the other three surfaces. The edges, as stated, are repeated six, the corners four times. In reading the triangles it must be remembered that the numbers along the edges refer to the letters that stand at the acute angles. For the inner shells one or two of the missing letter are to be added, and the remainder of the n things are then of the letter that stands at the right angle.

The general formula for calculating the number of combinations corresponding to any interior cell is $n!/A!B!C!D!$. We need only consider in each case the typical triangle ABC+tD. Since for the first tetra, D=1, the general formula becomes

$$\frac{n!}{B!C!1! [n - (B+C+1)]!} \text{. If we give various values to B and}$$

C, and put them in their proper places in the triangle, we obtain Table IX. It is apparent that this table has all the same regularities as Table V, so that we could here also obtain the interior terms from the edge terms of either columns, lines or diagonals, by determining proper coefficients. But we do not yet know the edge terms. These must themselves be derived somehow. If in imagination we follow any diagonal of the tetra out beyond the latter to where it pierces the surface of the pyramid, we shall find that it ends in a term that is suitable for our calculation. This feat of the imagination may not seem so easy, but the following plan may help.

Suppose Table V composed of a horizontal layer of cubes. Then Table IX, also composed of a horizontal layer of cubes, is to be set down on top of it, so that its first term lies upon the second term,

A	1	2	3	4	5	6	B
1	$\frac{n!}{1! 1! 1! (n-3)!}$	$\frac{n!}{2! 1! 1! (n-4)!}$	$\frac{n!}{3! 1! 1! (n-5)!}$	$\frac{n!}{4! 1! 1! (n-6)!}$	$\frac{n!}{5! 1! 1! (n-7)!}$	$\frac{n!}{6! 1! 1! (n-8)!}$	
2	$\frac{n!}{1! 2! 1! (n-4)!}$	$\frac{n!}{2! 2! 1! (n-5)!}$	$\frac{n!}{3! 2! 1! (n-6)!}$	$\frac{n!}{4! 2! 1! (n-7)!}$	$\frac{n!}{5! 2! 1! (n-8)!}$		
3	$\frac{n!}{1! 3! 1! (n-5)!}$	$\frac{n!}{2! 3! 1! (n-6)!}$	$\frac{n!}{3! 3! 1! (n-7)!}$	$\frac{n!}{4! 3! 1! (n-8)!}$			
4	$\frac{n!}{1! 4! 1! (n-6)!}$	$\frac{n!}{2! 4! 1! (n-7)!}$	$\frac{n!}{3! 4! 1! (n-8)!}$				
5	$\frac{n!}{1! 5! 1! (n-7)!}$	$\frac{n!}{2! 5! 1! (n-8)!}$					
6	$\frac{n!}{1! 6! 1! (n-8)!}$						

TABLE IX.

Formulas for calculating Typical Triangle ABC+1D of First Inner Pyramidal Shell.

General Formula $\frac{n!}{B(C!)^n [n-(B+C+1)]!}$
 Triangle is numbered from 1 to $n-3$. Table ends when
 $n=B+C+1$.

second column, of Table V. The latter now twice repeated, but with one line removed, is to be supposed set up on edge so as to enclose the right angle of IX with two vertical walls. Then it

is not difficult to see that the diagonal of the first term of IX pierces these back walls in the third term of the second line. The next diagonal of IX, of course, meets the fourth term, etc. By

A	1	2	3	4	5	6	7	8	B
1	2	3	4	5	6	7	8	9	Natural Nos.
2	3	6	10	15	21	28	36		Triangular Nos.
3	4	10	20	35	56	84			Pyramidal Nos.
4	5	15	35	70	126				Etc.
5	6	21	56	126					
6	7	28	84						
7	8	36							
8	9								
C									

TABLE X.

Coefficients for Calculating Typical Triangle of First Inner Pyramidal Shell.

Table is of Indefinite Extent.

A	1	2	3	4	5	6	7	B
1	$2 \times$ 360	$3 \times$ 840	$4 \times$ 1260	$5 \times$ 1260	$6 \times$ 840	$7 \times$ 360	$8 \times$ 90	
2	$3 \times$ 840	$6 \times$ 1260	$10 \times$ 1260	$15 \times$ 840	$21 \times$ 360	$28 \times$ 90		
3	$4 \times$ 1260	$10 \times$ 1260	$20 \times$ 840	$35 \times$ 360	$56 \times$ 90			
4	$5 \times$ 1260	$15 \times$ 840	$35 \times$ 360	$70 \times$ 90				
5	$6 \times$ 840	$21 \times$ 360	$56 \times$ 90					
6	$7 \times$ 360	$28 \times$ 90						
7	$8 \times$ 90							
C								

TABLE XI.

Method of Calculating Triangle ABC of Diag. 2, Table VII, from Second Line of Surface Triangle and Coefficients of Table X.

comparison of the two tables it is in fact seen that the factor $n!/[n - (B+C+1)]!$, which is constant along the diagonals of IX, is the same in the surface terms in which they end. It is only necessary then to determine the proper coefficients. This can be done as before by factoring the general expression, thus:

$$\frac{n!}{B! C! 1! [n - (B+C+1)]!} =$$

$$\frac{n!}{(B+C)! 1! [n - (B+C+1)]!} \times \frac{(B+C)!}{B! C!} = {}_n C_{B+C} \times {}_{B+C} C_B$$

Giving B and C their various values as before, we obtain Table X of the coefficients. It is seen at once that this table is exactly the same as Table VI for the surface triangle, except that the first line and column are omitted. The calculation of these inner terms hence becomes extremely simple, and may be reduced to the following rule.

The m th line of the first tetra is derived from the 2d line of the surface triangle by discarding $m+1$ terms, and multiplying the remaining terms by the $(m+1)$ -hedroidal numbers in order, beginning with the second.

A similar investigation will lead to a similarly simple result for the second tetra, which may be reduced to the following rule:

The m th line of the second tetra is derived from the 3d line of the surface triangle, by discarding $m+3$ terms, and multiplying the remaining terms by the $(m+2)$ -hedroidal numbers in order, beginning with the third.

Similar rules may be derived for the succeeding tetras, but if we call t the number of the tetra we may combine them all in one general rule as follows:

The m th line of the t th tetra is derived from the $(t+1)$ th line of the surface triangle by discarding $2t+m-1$ terms and multiplying the remaining terms by the $(t+m)$ -hedroidal numbers in order, beginning with the $(t+1)$ th.

This rule is general not only for all the inner tetras, but by putting t in it equal to 0 it reduces to the rule previously given for the surface triangle, which thus may be considered as the 0-tetra. This one rule hence covers all cases up to the present.

If we construct a series of pyramids, like that of Table VIII, for the successive values of n from 0 up, but give each a thickness of one cell in the direction of the fourth dimension, and pile the successive pyramids so that their A vertices are adjacent to each other in the direction of this dimension, then we shall obtain the four-dimensional arithmetical pyramid. Each three-dimensional pyramid will be a slice of the four-dimensional one, perpendicular to its fourth-dimensional axis, just as each two-dimensional diagram of Fig. 1 is a slice of the three-dimensional pyramid. Each cubical cell will now acquire a thickness equal to its edge in the direction of the fourth dimension and so become a four-dimensional cube, or teseract as it is sometimes called. The whole system will of course contain all classes of combinations up to four variations.

II.

In Part I we have dealt with the combinations of any number of things, each capable of 1, 2, 3 or 4 variations, and found that all possibilities could be represented by tables, having respectively 0, 1, 2 and 3 dimensions, viz., by the point, line, triangle and triangular pyramid. In each case we required a table of $k-1$ dimensions. Hence if we allow more than four variations we must, by the same rule, step out into space of higher dimensions, making use in each case of a $(k-1)$ -dimensional pyramid.

Let us first take the case of $k=5$. Call the variations A, B, C, D and E. By reasoning exactly analogous to that of the case $k=4$, it is clear that from every ABCD cell of the three-dimensional pyramid can be developed a series of new cells equal to the number of A in that cell, by exchanging successively the A's for E's. The only proper place to put these new cells is to build them out from the respective ABCD cells from which they were developed, in the direction of the *fourth* dimension. Because of the regularly diminishing number of the A's in the cells, in passing outward from the A vertex toward the BCD plane, the new solid developed will have the form of a four-dimensional pyramid, analogous to the three-dimensional pyramid previously described. We shall call it a *pentahedroid*, or a *penta* for short, though it is really right-angled instead of equilateral. This penta, as shown by the fifth line of the arithmetical triangle, is bounded by 5 corners, 10 edges, 10 triangular surfaces and five tetrahedra, all enclosing an interior four-dimensional space. These configurations will carry respectively the classes of 1, 2, 3, 4 and 5 variations. The typical triangles of the classes of 1, 2, 3 and 4 variations will be exactly the same as before, except for the different number of repetitions. The five bounding tetras will have interior shells exactly the same as those of diagrams 2 and 3 of Table VIII, and these being independent of one another will be repeated in entirety five times. The 20 surface triangles of the 5 tetras however coincide in pairs, reducing to 10; the 30 edges coincide in threes, reducing to 10; the 20 corners coincide in fours, becoming 5, as already stated. In other words 4 instead of 3 edges now radiate from every vertex, 3 instead of 2 planes from every edge, while every plane divides 2 adjacent tetras from each other.

These 10 surface triangles and the interior shells of the 5

bounding tetras constitute the surface or zero shell of the pentahedroid. The interior space can be considered as before to be made up of concentric pentahedroidal shells, each one cell in thickness in the direction of the fourth dimension. Each such shell will be exactly similar to the surface shell. It will have the same number and kind of boundaries, and can hence be represented in just the same way, viz., by 10 surface triangles, and the interior shells of the 5 bounding tetrahedra. The latter will be called: the first inner tetra shell of the first inner penta shell, second tetra of first penta, etc.

Each inner penta shell will have five less cells on a side than the next outer shell. There will therefore be $n/5$, neglecting remainder, such inner shells. Each will contain one more of each of the two missing letters. The typical triangles, which we shall call the surface triangles of the inner pentas, will be lettered and numbered as follows:

1st penta,	ABC + 1D + 1E	1 to $n - 4$
2d " "	ABC + 2D + 2E	2 to $n - 2 \cdot 4$
3d " "	ABC + 3D + 3E	3 to $n - 3 \cdot 4$
p th " "	ABC + p D + p E	p to $n - 4p$

The tetras of the inner pentas will be lettered and numbered as follows:

NAME OF TETRA	LETTERING	NUMBERING	NO. OF INNER TETRA SHELLS
1st tetra of 1st penta	ABC + 2D + 1E	2 to $n - 3 - 4$	$\frac{n-5}{4}$
2d " " 1 "	ABC + 3D + 1E	3 to $n - 2 \cdot 3 - 4$	
t th " " 1 "	ABC + $(t+1)$ D + 1E	$t+1$ to $n - 3t - 4$	
1st " " 2d "	ABC + 3D + 2E	3 to $n - 3 - 2 \cdot 4$	$\frac{n-2 \cdot 5}{4}$
2d " " "	ABC + 4D + 2E	4 to $n - 2 \cdot 3 - 2 \cdot 4$	
t th " " "	ABC + $(t+2)$ D + 2E	$t+2$ to $n - 3t - 2 \cdot 4$	
t th " " p th "	ABC + $(t+p)$ D + p E	$t+p$ to $n - 3t - 4p$	$\frac{n-5p}{4}$

Of course for the other triangles all combinations of the five letters will be taken. This case for $n = 10$ is worked out in Table XII. Similar tables can be made for other values of n .

Comparing the system of rational numbers, in order of magnitude, with the points of a straight line L , we see that, if any origin be taken on L and a fixed unit of measurement, to any rational number a can be constructed a corresponding point; but there are points (those determined by incommensurable lengths measured from o) to which no rational numbers correspond. Thus we can say that " L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals."⁴ So if, as we wish,⁵ all phenomena in the straight line are also to be followed out arithmetically⁶ R must be refined by the creation of new numbers, and the domain of numbers raised to the same completeness—or "continuity"—as the straight line.

"The way in which irrational numbers are usually introduced is connected with the concept of extensive magnitude—which itself is nowhere rigorously defined—and explains number as the result of the measurement of one such magnitude by another of the same kind.⁷ Instead of this I demand that arithmetic shall be developed out of itself. That such connections with non-arithmetical notions have furnished the immediate occasion for the extension of the number-concept may, in general, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign connections into arithmetic, the science of numbers. Just as negative and fractional rational numbers must and can be formed by a new creation, and as the laws of operation with these numbers must and can be reduced to the laws of operation with posi-

⁴ *Stetigkeit*, p. 9; *Essays*, p. 9. ⁵ "Was doch der Wunsch ist," *ibid.*

⁶ Cf. *Stetigkeit*, pp. 5-6, 10; *Essays*, pp. 4, 10.

⁷ "The apparent advantage of this definition of number in point of generality vanishes the moment we think of complex numbers. In my view, the conception of the ratio to one another of two magnitudes of the same kind can be clearly developed only after the irrational numbers have been introduced."

tive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. There only remains the question as to how to do this."⁸

Now the essence of this "continuity" of L was found by Dedekind⁹ after long meditation to be: If all the points of L fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which generates this division. This, as Dedekind emphasized, will probably be considered as evidently true by every one; it cannot be proved, but is an axiom by means of which we first recognize the line of its continuity. If space has a real existence, it need not necessarily be continuous; many of its properties would remain the same if it were discontinuous¹⁰; and if we knew that it was discontinuous, nothing could prevent us, if we wished, making it continuous in thought by filling up its lacunae. Another simple logical transformation of the above axiom is not so obvious: there is one and only one point (of the first class) which is on the extreme left of the first class, or one and only one of the second class on the extreme right of the second class, but not both.

⁸ *Steigkeits*, p. 10; *Essays*, pp. 9-10.

⁹ *Stetigkeit*, p. 11; *Essays*, p. 11. This axiom has been frequently misunderstood; thus L. Couturat (*De l'infini mathématique*, Paris, 1896, p. 416) stated it: "If all the quantities of a kind can be divided into two classes such that all the quantities of the one precede (or follow) all those of the other, there exists a quantity of this kind which *both* follows all those of the inferior class *and* precedes all those of the superior class." Russell, in a review (*Mind*, Vol. VI, 1897, p. 117), rightly pointed out the mistake in this wording but wrongly advanced the same criticism against Dedekind's own axiom (*The Principles of Mathematics*, Vol. I, Cambridge, 1903, p. 279). In fact, we do not need, as Russell presumed, a "point left over to represent the section"; and Russell's (second) "emendation" (pp. 279-280) is Dedekind's original axiom.

¹⁰ An example of this was given in the preface of *Was sind und was sollen die Zahlen?* (*Essays*, pp. 37-38). Choose any three points A, B, C, which do not lie in a straight line and which are such that the ratios of their distances AB, AC, BC are algebraic numbers; and regard as present in space only those points M for which the ratios of AM, BM, and CM to AB are algebraic numbers. The space consisting of the points M is everywhere discontinuous, but yet in it all the constructions in Euclid's *Elements* can be carried out just as well as in a continuous space.

The purely arithmetical definition of new numbers among those of the system R so as to make it a continuous system was now brought about on a basis analogous to that of the above axiom. Any rational number a brings about a division of the system R into two classes A_1, A_2 , such that any number of A_1 is smaller than any number of A_2 ; a is either the greatest of A_1 or the least of A_2 . If now we have any division of R into classes A_1, A_2 , such that any member of A_1 is smaller than any member of A_2 , we call such a division a "section" or "cut" (*Schnitt, coupe*), and denote it by (A_1, A_2) . We can then say that any rational number a generates a section, or strictly speaking two sections (which, however, we will not regard as essentially different).¹¹ But there are an infinity of sections—such as that where A_1 consists of all the rational numbers r such that $r^2 < D$ is a positive non-square integer, and A_2 of the rest—which are not generated by rational numbers,—that is to say, neither has A_1 a maximum nor A_2 a minimum; and in this consists the incompleteness or discontinuity of R. Now, whenever we have a section (A_1, A_2) generated by no rational number, we create (*erschaffen*) a new, an "irrational number," which we regard as completely defined by the section (A_1, A_2) and is said to generate it.¹²

By comparing two sections, (A_1, A_2) and (B_1, B_2) , as to the inclusion or not of any term of A_1 in B_1 , or *vice versa*, we arrive at a basis for determining the order of any two real (rational or irrational) numbers α and β as symbolized by

$$\alpha = \beta, \alpha > \beta, \text{ or } \alpha < \beta,¹³$$

and also definitions of new sections whose generators may be represented by

¹¹ *Stetigkeit*, p. 12; cf. *Essays*, p. 13.

¹² *Stetigkeit*, p. 14; *Essays*, p. 15.

¹³ *Stetigkeit*, pp. 15-19; *Essays*, pp. 15-21.

$\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$ and α^β ,

may be given.¹⁴

We will now indicate the use of the-conception of a section to prove the theorems on limits mentioned above.¹⁵ A variable x is said to have a fixed limiting value α , when $x - a$ ultimately sinks, numerically speaking, below any positive, non-zero, number; and our first theorem is that, if x increases continually, but not beyond all values, it approaches a *definite* limit. By the supposition, we have numbers α_2 such that we always have $x < \alpha_2$; denote the totality of these numbers by A_2 , and that of the other real numbers by A_1 . Any member (α_1) of A_1 has the property that in course of the process $x \leq \alpha_1$, and so every member of A_1 , is smaller than any member of A_2 , so that (A_1, A_2) is a section. Its generator (α) is either the greatest in A_1 or the least in A_2 ; the former cannot be the case, because x never ceases to increase. Thus α is the least member of A_2 , and it is a limit of the x 's, for, whatever member of A_1 the number α_1 may be, we ultimately have $\alpha_1 < x < \alpha$.

Still more often used is the equivalent of this theorem: If, in the process of variation of x , for any positive δ (however small) a corresponding place can be given from which one x varies by less than δ , then x approaches a limiting value. This can easily be derived from the foregoing theorem, or directly, as we do here, from the principle of continuity.

If $x = a$ at the instant referred to in the theorem, ever afterwards $x > a - \delta$ and $x < a + \delta$. On this fact we found a double separation of the system of real numbers. Put every number α_2 such that, in the course of the process, we have $x \leq \alpha_2$, in a class A_2 , and let A_1 consist of all the other numbers; so that, if α_1 is such a number it will happen

¹⁴ *Stetigkeit*, pp. 19-22; *Essays*, pp. 21-24.

¹⁵ *Stetigkeit*, pp. 22-24; *Essays*, pp. 24-27.

infinitely often, however far the process may have progressed, that $x > a_1$. Since any a_1 is less than any a_2 , there is a definite generator α of the section (A_1, A_2) , which we will call the upper limiting value of x . Similarly, a second section (B_1, B_2) of the system of real numbers is brought about by x , if any number β_1 (such as $a - \delta$) such that in the course of the process $x \leq \beta_1$ is put in B_1 ; and the generator β is called the lower limit of x . The two numbers α and β are also evidently characterized by the property that, if ϵ is taken positive and arbitrarily small, we always have $x < \alpha + \epsilon$ and $x > \beta - \epsilon$, but never finally $x < \alpha - \epsilon$ and never finally $x > \beta + \epsilon$. Now, two cases are possible: if α and β are different from one another (so that $\alpha > \beta$), x oscillates, and suffers, however far the process may have progressed, variations whose amount exceeds $(\alpha - \beta) - 2\epsilon$. But the original supposition, which is now first used, excludes this, and so there only remains the case $\alpha = \beta$; and we see that x approaches the limiting value α .

Dedekind remarked¹⁶ that, while the lengthiness in the definitions of the elementary operations can partly be overcome by the use of auxiliary concepts such as that of an "interval" (a system of rational numbers such that, if a and a' are any members of it, all the numbers between a and a' are also members of it)¹⁷ and of its limits, yet "still lengthier considerations seem to loom up when we wish to transfer the innumerable theorems of the arithmetic of rational numbers, as, for example, the theorem $(a + b)c = ac + bc$, to any real numbers. However, this is not so, for we soon convince ourselves that here all reduces to proving that the arithmetical operations themselves have a certain continuity. What I mean by this I will put in form of a general theorem: If the number λ is the result of a calcula-

¹⁶ *Stetigkeit*, pp. 20-22; *Essays*, pp. 22-24.

¹⁷ Both the classes of any section are "intervals."

tion undertaken with the numbers $\alpha, \beta, \gamma, \dots$, and if λ lies inside the interval L , then intervals A, B, C, \dots , inside which $\alpha, \beta, \gamma, \dots$, respectively lie, can be given such that the result of the same calculation in which $\alpha, \beta, \gamma, \dots$ are replaced by any numbers of A, B, C, \dots respectively, is always a number lying inside L . The forbidding clumsiness, however, which marks the enunciation of such a theorem convinces us that here something must be done to aid language. This is done in the most satisfactory way by introducing the concepts of *variable magnitudes*, *functions*, and *limiting values*; and indeed the most convenient thing is to base the definitions of the simplest arithmetical operations on these concepts, but this cannot be carried farther here.”¹⁸

II.

The last few words contain an indication of the fundamental concepts upon which Dedekind’s theory of integers was based. The notion of an aggregate or “system” of things is, of course, the most fundamental, and also we utilize, in counting, the capability of the mind to *refer* things to things, to let a thing *correspond* to a thing, or to *image* (*abzubilden*) a thing by a thing. Without this capability no thought is possible, and on this single, but quite indispensable, foundation must, in Dedekind’s view, the whole science of numbers be erected.¹⁹ This idea of

¹⁸ *Stetigkeit*, pp. 21-22; cf. *Essays*, pp. 23-24.

¹⁹ In the eleventh appendix of Dedekind’s edition of Dirichlet’s *Vorlesungen über Zahlentheorie* (3d ed., 1879, § 163, p. 470), Dedekind said: “It happens very frequently in mathematics and other sciences that, if we have a system Ω of things or elements ω , every definite element ω is replaced according to a certain law by a definite element ω' corresponding to it. We are accustomed to call such an act a substitution and say that by this substitution the element ω passes over into the element ω' and the system Ω into a system Ω' of elements ω' . The expression of this is somewhat more convenient if we conceive this substitution as a transformation (*Abbildung*) of the system Ω .” To this he added the note: “On this ability (*Fähigkeit*) of mind to compare a thing ω with a thing ω' , or to refer ω to ω' , or to let ω correspond to ω' , without which thought is impossible, rests, as I will try to prove in another place, the whole science of numbers.”

correspondence is the idea of functionality or, in other words, of establishing a *relation* between things.

Dedekind's views on the nature of numbers may be expressed as follows. Arithmetic, including Algebra and Analysis, "is a part of logic, and the number-concept is quite independent of the notions or intuitions of space and time, and is an immediate consequence of the pure laws of thought." Toward the beginning of his *Stetigkeit*,²⁰ he wrote: "I regard the whole of arithmetic as a necessary or at least natural consequence of the simplest arithmetical act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers, in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already formed individual to the consecutive new one to be formed. The chain of these numbers forms even by itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any repetitions we wish of the above mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been and whatever comparisons or analogies with experience or intuition may have led us, it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act. Thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater perfection. Numbers are free creations of the human mind; they serve as a

²⁰ Pp. 5-6; cf. *Essays*, p. 4.

means to grasp the difference of things more easily and distinctly. Only by means of the purely logical structure of the science of numbers and the continuous number-region obtained in it are we in a position accurately to investigate our notions of space and time, by referring them to this number-domain created in our mind."

Dedekind had the intention of showing the development of the conception of the natural (integral) numbers from the purely logical conceptions of aggregate and "representation" (*Abbildung*), before the publication (1872) of his work on continuity, but it was only after the appearance of this work that, from 1872 to 1878, he wrote out a sketch of his system containing all its essential ideas, and showed it to and discussed it with many mathematicians. In 1887 a careful exposition was carried out and published in the next year under the title *Was sind und was sollen die Zahlen?*²¹ The motive for the publication was the appearance of the essays of Kronecker and von Helmholtz. His own work, as he said, though similar in many respects to those essays, was in its foundations essentially different, and he had formed his own view "many years before and without influence from any side."

Dedekind regarded the maxim that "in science anything which can be proved is not to be accepted without proof"²² as unfulfilled even in the most recent methods of laying the foundations of arithmetic. And Dedekind's answer to this want was one of the first examples of that tendency of modern mathematics to extend exactness of treatment to the very principles, that has been gradually carried out by mathematical logicians like Frege, Peano and Russell.

As we should expect, the tract at first excited the derision of those unperceiving mathematicians who thought

²¹ Brunswick, 1888; second unaltered edition, 1893 [prefaces dated Oct. 5, 1887 and Aug. 24, 1893]; *Essays*, pp. 31-115.

²² *Was sind und was sollen die Zahlen?*, p. vii; cf. *Essays*, p. 31.

that Dedekind was merely taking an unnecessarily long time to prove obvious things like the commutative law in arithmetic. That such things seem to be immediately obvious will at once be granted, but the logical problem which interested Dedekind and many others since about the middle of the nineteenth century was whether or not such theorems are logically implied by those (logical) principles which hold for all true thought without exception, and are not of merely empirical validity. If we are in sympathy with efforts to solve the problems of the nature of our knowledge, we ought not to complain that the detailed writing out of logical steps takes up a large space. Besides, such a complaint is irrelevant.

Dedekind considered what he called "systems," which are what logicians call "classes" and mathematicians now usually call "aggregates," and then the idea of a correspondence of the elements of a system with elements of another system or with one another. He viewed such a correspondence as a "transformation"; and, when he came to consider "similar [or one-to-one] transformations of a system into a part of itself," he arrived at defining an "infinite" system²³ and thus fell upon much the same ideas that Georg Cantor independently did.²⁴ A special infinite system is the "simply infinite system" N which is such that there exists a similar transformation φ such that $\varphi(N)$ is a part of N , and N is the common part of all systems S which contain a definite element of N which is not of $\varphi(N)$, and for which $\varphi(S)$ is a part of S .²⁵ We can see without much difficulty that N consists of an element a , its transform a' , the transform a'' of a' , and so on; but it is to be noticed that Dedekind defines his infinite systems as wholes and does not use the vague words "and so on."

²³ *Essays*, pp. 63, 41-42.

²⁴ *Contributions to the Founding of the Theory of Transfinite Numbers*, Chicago and London, 1915, p. 41.

²⁵ Cf. *Essays*, pp. 67, 56-58.

The ordinal numbers then appear as mental abstractions from such systems as N ,²⁶ the theorem of complete induction is proved for them,²⁷ and the various other fundamental arithmetical concepts and theorems established. In particular, Dedekind considered cardinal numbers to be logically subsequent to ordinal numbers.²⁸

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²⁶ *Ibid.*, p. 68.

²⁷ *Ibid.*, pp. 69-70, 60-62, 32-33, 42-43.

²⁸ *Ibid.*, pp. 109-110, 32.

CRITICISMS AND DISCUSSIONS.

THE ARITHMETICAL PYRAMID OF MANY DIMENSIONS

AN EXTENSION OF PASCAL'S ARITHMETICAL TRIANGLE TO
THREE AND MORE DIMENSIONS, AND ITS APPLI-
CATION TO COMBINATIONS OF MANY
VARIATIONS.

I.

In 1665 Pascal wrote his *Traité du triangle arithmétique* and showed that the system of numbers there developed, the so-called *figurate* numbers, had many remarkable properties. The most useful of these, and for our present purposes the most important, is the fact that this table gives the value of the expression ${}_nC_r$, for all positive integral values of n and r (including 0). The expression ${}_nC_r$ means the number of combinations of n things taken r at a time.

It is also written $\binom{n}{r}$, and is equal to

$$\frac{n(n-1)(n-2)(n-3)\dots(n-r+1)}{r!}$$

or $n!/r!(n-r)!$, in which $r!$ is read "factorial r " and denotes the product of all the integral numbers from 1 to r inclusive. The appropriate solution for any given values of n and r is to be found in the n th line and r th column of the arithmetical triangle. See Table I.

Now ${}_nC_r$ refers to things, each of which is capable of two and only two variations, such as coins that may fall either heads or tails. But frequently we have to do with things subject to more than two variations, such as a number of signal lights each showing several colors, or a number of dice which may fall on any one of their six faces. The solutions of such cases are not to be found in the arithmetical triangle, though in every case they can be shown to be

Names of the Columns	Natural Nos.	Triangular Nos.	Pyramidal Nos.	Pentahedroid Nos.	Hexahedroid Nos.	Septahedroid Nos.	Octahedroid Nos.	Nonahedroid Nos.	Dekahedroid Nos.	Undekahedroid Nos.
Boundaries of the Figures	Corners	Edges	Sur-faces	Tetra-he-dra	Penta-he-dro-ids	Hexa-he-dro-ids	Septa-he-dro-ids	Octa-he-dro-ids	Nona-he-dro-ids	Deka-he-dro-ids
	0	1	2	3	4	5	6	7	8	9
	0	1								1
Point	1	1	1							2
Line	2	1	2	1						4
Triangle	3	1	3	3	1					8
Tetrahedron	4	1	4	6	4	1				16
Pentahedroid	5	1	5	10	10	5	1			32
Hexahedroid	6	1	6	15	20	15	6	1		64
Septahedroid	7	1	7	21	35	35	21	7	1	128
Octahedroid	8	1	8	28	56	70	56	28	8	256
Nonahedroid	9	1	9	36	84	126	126	84	36	512
Dekahedroid	10	1	10	45	120	210	252	210	120	45
Combinations	nC_0	nC_1	nC_2	nC_3		nC_r		nC_{n-2}	nC_{n-1}	nC_n
General Formulas	$\frac{n!}{0! (n-0)!} \frac{n!}{1! (n-1)!} \frac{n!}{2! (n-2)!} \frac{n!}{3! (n-3)!}$				$\frac{n!}{r! (n-r)!}$			$\frac{n!}{(n-2)! 2!}$	$\frac{n!}{(n-1)! 1!}$	$\frac{n!}{n! 0!} = 2^n$

TABLE I. The Arithmetical Triangle.
See also Table VI for another arrangement.

the product of two or more numbers there to be found. So far as the writer is aware no systematic method of selecting the proper factors has yet been given.

In the case of two variations, for any given value of n there will be $n+1$ classes, obtained by giving r successively all integral values from 0 to n . In any class r is the number of one kind present, $n-r$ the number of the other. These can all appropriately be arranged along a straight line. In fact the complete set of solutions is to be found in the n th line of the arithmetical triangle. But if the n things are capable of more than two variations—if for example they may be A's, B's, C's, D's, etc.—then a much larger number of classes arises; for to any one of these letters may be assigned in turn all the integers from 0 to n , and all vary independently. These classes cannot be so simply arranged, and the task of obtaining all of them and calculating the number of combinations for each becomes very complicated. Some systematic method must be adopted to insure exhaustive enumeration.

The object of the present paper is to show how these cases of many variations may be appropriately arranged in more-dimensional tables, so as to develop with certainty all possible classes, and show their proper relations to one another, and also to show how the arithmetical triangle may likewise be extended to more dimensions, and thus provide means of readily finding the number of combinations corresponding to each class. The method is somewhat complicated to explain, but easy to operate. We shall begin by describing a few of the many remarkable properties of the arithmetical triangle, such as will be useful to us, and then take up in turn its extension to 2, 3, 4, . . . , k variations.

All the numbers of the arithmetical triangle can of course be calculated from the general formula already given, $n!/r!(n-r)!$. But the table can also be much more simply produced by a process of successive addition as follows: Beginning with 1, below any line write the same line moved one place to the right and add. The result is the next line. The process is shown in Table II.

From the mode of development it is apparent that the differences of any column are to be found in the next column to the left. Any column is therefore an arithmetical series of the r th order, whose r th differences are constant and equal to 1. The table is in fact the complete system of all arithmetical series whose final differences are 1. Conversely each number gives the sum of all the

preceding numbers of the next column to the left, or the sum of any two numbers in the same line is found immediately below the right-hand one.

Each line gives the binomial coefficients in order for the exponent corresponding to the number of the line, for these coefficients are also given by the formula $\binom{n}{r}$. The sum of all the numbers of any line is 2^n .

The columns have been given special names because of certain properties they possess. The zero column is composed only of units. The first column contains the natural numbers. The second contains the triangular numbers, so called because they give the number of units that can be arranged in a triangle, having suc-

Line Zero	1
	1
Line One	<u>1 1</u>
	1 1
Line Two	<u>1 2 1</u>
	1 2 1
Line Three	<u>1 3 3 1</u>
	1 3 3 1
Line Four	<u>1 4 6 4 1</u>

TABLE II. Method of Constructing Arithmetical Triangle.

sively 1, 2, 3, 4, . . . units on a side. The third column contains the pyramidal numbers, so called because they give the number of units that can be piled like cannon balls in the form of a triangular pyramid or tetrahedron, having successively 1, 2, 3, 4, . . . units on a side.

The remaining columns have as yet received no special names, but they might appropriately be named after the succeeding higher-dimensional pyramids, since they similarly give in turn the numbers of units that can be arranged in the form of these higher pyramids, having successively 1, 2, 3, 4, . . . units on a side. We shall call the latter, after Stringham, successively, the 4-dimensional pentahedroid, the 5-dimensional hexahedroid, the 6-dimensional septahedroid, etc., in general the $(k-1)$ -dimensional k -hedroid, and name the columns

after them as shown in Table I. Thus the tetrahedron becomes a 4-hedroid, the triangle a 3-hedroid, the line a 2-hedroid, the point a 1-hedroid, and the corresponding columns the 4, 3, 2, 1-hedroidal numbers respectively.

Most useful for our subsequent purposes however is the fact that the arithmetical triangle gives complete specifications for the construction of any of these higher pyramids. Thus the n th line gives in order the number of 0, 1, 2, 3, ..., $(n-1)$ -dimensional boundaries of the $(n-1)$ -dimensional n -hedroid. We have only to read the line designating the succeeding numbers in turn as so many corners, edges, surfaces, tetrahedra, etc., as indicated in Table I. Thus we may read:

First line: One point has 1 corner or 0-space boundary.

Second line: One line has 2 corners or ends, and 1 edge or interior 1-space.

Third line: One triangle has 3 corners, 3 edges, and 1 interior 2-space or surface.

Fourth line: One tetrahedron has 4 corners, 6 edges, 4 surfaces, and 1 interior 3-space.

Fifth line: One pentahedroid has 5 corners, 10 edges, 10 triangular surfaces, and 1 interior space of four dimensions.

Similarly the remaining lines may be read.

It may be noted that as the line lies between its ends, the triangle within its edges, the 3-space of the tetrahedron inside its bounding 2-space surfaces, so the 4-space of the pentahedroid is *inside* its bounding 3-space tetrahedra. Similarly with the higher pyramids. The 5-space is inside the 4-space, the 6- inside the 5-, etc. We get to higher and higher degrees of insideness.

As we shall use these higher pyramids to represent our combinations of many variations, it is important to know how they are constructed.

We may now proceed to our task of applying the arithmetical triangle to the cases of two and more variations. Calling k the number of variations, it will be found in every case that a $(k-1)$ -dimensional k -hedroidal table will be required, the total number of classes is given by the $(n+1)$ th k -hedroidal number, while the sum of all the combinations is k^n . This gives a valuable check on the correctness of the work. The variations we shall call A, B, C, D, etc.

Let $k = 2$. The complete set of solutions for this case, as has already been stated, is to be found in the n th line of the arithmetical

triangle. They form therefore a linear table, as shown for $n=10$, in Table III. The second half of any such line is always the reverse of the first half, so that there are only $(n+2)/2$, disregarding remainder, different numbers in it.

	0	1	2	3	4	5	6	7	8	9	10	B
A	10	9	8	7	6	5	4	3	2	1	0	
	1	10	45	120	210	252	210	120	45	10	1	

TABLE III. All Classes Combinations of 10 Things each susceptible of 2 Variations, such as 10 Coins, that may fall Heads or Tails.

Number of classes $= n+1 = 11$. Sum of combinations $= 2^{10} = 1024$.

Note. First line shows number of B's present, second line number of A's.

Let $k=3$. Each thing may now be an A or a B or a C. The additional classes possible may be derived from the previous case in the following way. To each class of A and B, as shown in Table III, additional classes may be formed, by keeping the B's constant, and exchanging the A's one by one for C's. The number of new classes thus formed will in each case be equal to the number of A's contained in the original cell. Thus the first cell gives 10, the second 9, the third 8, etc. new classes. The only appropriate way to arrange the new classes is in columns vertically under the cells from which they were developed, according to descending values of A, or ascending values of C. In this way the triangular Table IV is obtained, for the value of $n=10$. It may be noted that there will be three classes where the n things are all of one kind, as all A's, all B's, or all C's. These are placed at the corners, or 0-dimensional boundaries of the triangle. There will be three sets of classes where the things are AB, BC, or AC. Each of these three sets is a duplicate of Table III, and they are located on the three edges of the triangle. The remaining classes in which all three variations appear are all located in the interior of the triangle, or in 2-space.

Further the triangle may be divided, as shown by the dash lines of Table IV, into a series of similar concentric triangular shells. The first interior shell contains all classes where one of the letters or variations appears each time only once. The second interior shell contains all classes where one of the letters appears always twice, the third three, etc. There will always be $n/3$, disregarding remainder, of these interior shells. It may further be noted that in each line,

column and diagonal, the second half is the reverse of the first half. The same numbers are hence oft repeated. In fact in Table IV there are only 14 different numbers, while if we deduct the 6 taken direct from Table III there are only 8 new numbers to calculate. These are shown enclosed in heavy lines in the table. It is easily seen that they cover approximately $\frac{1}{6}$ of the total area of the triangle, hence may be calculated for any n from $n^2/12$, taking the

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

Note. The first line shows number of B's present, left-hand column the number of C's. The remaining $n - (B + C)$ are A's.

Number of classes = 11th triangular number or $[(n + 1)(n + 2)]/2 = 66$.

Sum of all combinations = $3^{10} = 59,049$.

TABLE IV. Combinations of 3 Variations for $n = 10$.

nearest whole number. For the values of n from 1 to 15 the following are obtained as the new interior numbers to calculate:

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.$$

Nos. 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19.

The fact that for the first two cases no new numbers are obtained means that all the cells abut on the edges. When $n = 3$ we get 1 interior cell; when $n = 4$ we get 3 cells but all are filled with the same number, viz. 12, as the reader may verify, if he likes, by working out these simple cases. Hence the number of calculations to make is not very great unless n is large.

It will be noted of course that the numbers increase in size toward the geometric center of the triangle. Also from the method

of numbering, the B's are constant in any column, the C's along any line, and each increases in value from the right angle outward. The A's are constant along any diagonal and increase in value toward the right angle, where they are all A's. This method of lettering and numbering will be adhered to in all that follows, and the characteristics that depend upon it will naturally always recur.

The value of any interior number is given by the general formula

$$\frac{n!}{A!B!C!} = \frac{n!}{B!C![n-(B+C)]!}, \text{ since } A+B+C=n. \text{ Calling}$$

$0!=1$, as is customary, this formula will also apply to the edges and hence to all the numbers of the triangle. But there is a much easier way of deriving the appropriate numbers directly from the arithmetical triangle, which it is one of the objects of this paper to show. If in the above formula we give to A, B and C different values, and write them down in their proper places, we obtain Table V, below. Since, as already stated, the A's, B's and C's are constant respectively along the diagonals, columns and lines it is apparent that the expression $n!/A!B!C!$ can, in three ways, be divided into two factors, one of which is constant and the other variable, according as we choose the constant part along a diagonal, a column or a line. Thus,—

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!} \times \frac{1}{C![n-(B+C)]!} \quad (1)$$

$$\text{or } = \frac{n!}{C!} \times \frac{1}{B![n-(B+C)]!} \quad (2)$$

$$\text{or } = \frac{n!}{[n-(B+C)]!} \times \frac{1}{B!C!} \quad (3)$$

In every case the first factor is constant, the second variable. In (1) first factor is constant along a column, in (2) along a line, in (3) along a diagonal. Not much is gained by this, but if we multiply numerator and denominator in the three cases respectively by $(n-B)!$, $(n-C)!$, $(B+C)!$, we obtain,

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!(n-B)!} \times \frac{(n-B)!}{C![n-(B+C)]!} = {}^nC_B \times {}^{n-B}C_C \quad (4)$$

$$\text{or } = \frac{n!}{C!(n-C)!} \times \frac{(n-C)!}{B![n-(B+C)]!} = {}^nC_C \times {}^{n-C}C_B \quad (5)$$

$$\text{or } = \frac{n!}{(B+C)![n-(B+C)]!} \times \frac{(B+C)!}{B!C!} = {}^nC_{B+C} \times {}^{n-B-C}C_B \quad (6)$$

In all three cases now each factor is seen to be a figurate number, hence one to be found in the arithmetical triangle. Moreover

A	A	0	1	2	3	4	5	6	B
0	0	$\frac{n!}{0!0! [n-(0+0)]!}$	$\frac{n!}{1!0! [n-(1+0)]!}$	$\frac{n!}{2!0! [n-(2+0)]!}$	$\frac{n!}{3!0! [n-(3+0)]!}$	$\frac{n!}{4!0! [n-(4+0)]!}$	$\frac{n!}{5!0! [n-(5+0)]!}$	$\frac{n!}{6!0! [n-(6+0)]!}$	
1	1	$\frac{n!}{0!1! [n-(0+1)]!}$	$\frac{n!}{1!1! [n-(1+1)]!}$	$\frac{n!}{2!1! [n-(2+1)]!}$	$\frac{n!}{3!1! [n-(3+1)]!}$	$\frac{n!}{4!1! [n-(4+1)]!}$	$\frac{n!}{5!1! [n-(5+1)]!}$	$\frac{n!}{6!1! [n-(6+1)]!}$	
2	2	$\frac{n!}{0!2! [n-(0+2)]!}$	$\frac{n!}{1!2! [n-(1+2)]!}$	$\frac{n!}{2!2! [n-(2+2)]!}$	$\frac{n!}{3!2! [n-(3+2)]!}$	$\frac{n!}{4!2! [n-(4+2)]!}$			
3	3	$\frac{n!}{0!3! [n-(0+3)]!}$	$\frac{n!}{1!3! [n-(1+3)]!}$	$\frac{n!}{2!3! [n-(2+3)]!}$	$\frac{n!}{3!3! [n-(3+3)]!}$				
4	4	$\frac{n!}{0!4! [n-(0+4)]!}$	$\frac{n!}{1!4! [n-(1+4)]!}$	$\frac{n!}{2!4! [n-(2+4)]!}$					
5	5	$\frac{n!}{0!5! [n-(0+5)]!}$	$\frac{n!}{1!5! [n-(1+5)]!}$						
6	6	$\frac{n!}{0!6! [n-(0+6)]!}$							

TABLE V.
Formulas for Calculating Combinations of 3 Variations.

This table is of indefinite Extent, but for any given value of n comes to an end when $n = B + C$.

First factorial of denominator gives number of B's.
 Second " " " " " " C's.
 Third " " " " " " A's.

General formula: $\frac{n!}{B! C! [n-(B+C)!]}$

the first factor is not only constant but in (4) it is *equal* to the number that stands at the head (and foot) of the corresponding column, in (5) to the number that stands at either end of the corresponding line, in (6) to the number that stands at either end of the corresponding diagonal of the table to be calculated, while in each case the second factor for these terms becomes 1. By the first scheme then all the members of any column could be obtained by multiplying the first one by the successive values of the second factor of (4), obtained by giving B and C the proper values. Similarly by the second scheme all the terms of any line could be obtained from the first one, while by the third scheme all the terms of any diagonal from the end ones. Any one of the three schemes would produce the whole triangle, and it would seem natural to choose either of the first two. However because of the occurrence of $n-B$, and $n-C$, in these two schemes, making it necessary to assign a definite value to n before anything can be done, they do not lend themselves so readily to general treatment as the third scheme. We shall accordingly adopt the latter.

The proper values of the second factor, or coefficient as we shall call it, could of course be found in line B, or C, of column $B+C$ of the arithmetical triangle. But if we give different values to B and C and write the resulting numbers down in their proper places in the triangle, we obtain Table VI. It is at once seen that they follow a very regular order, being in fact nothing other than the arithmetical triangle itself, with each column pushed up to the top. This is the usual arrangement of the figurate numbers. The diagonals of the new table are the lines of the old.

The procedure of calculating a triangle then is as follows. As the first line write the n th line of the arithmetical triangle. Any interior number is then calculated by multiplying the number that stands at the end of its diagonal by the coefficient shown in Table VI. The process is shown in Table VII. Or the process may be described thus: Each successive line is derived from the first line by discarding each time one additional term and multiplying the remaining terms first by the natural, then by the triangular, then by the pyramidal, etc., numbers in order. Or we may sum the whole thing up in one general rule:

The m th line of a "surface triangle" is derived from the n th line of the arithmetical triangle by discarding $m-1$ terms and multi-

plying the remaining terms by the *m*-hedroidal numbers in order, beginning with the first one.

This rule applies to the outside edge as well as to the inner lines. The reason for calling the figure a *surface triangle* is because the surfaces of all our subsequent higher pyramids will be composed of such figures.

We have used for purposes of illustration $n = 10$. If we give to n other values we shall obtain similar triangles, smaller or larger according to the value of n . If we construct a series of these from $n = 0$ up, and pile them all up on top of each other with their A

A	0	1	2	3	4	5	6	7	8	9	10	11	12	B
0	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	2	3	4	5	6	7	8	9	10	11	12		Natural Nos.
2	1	3	6	10	15	21	28	36	45	55	66			Triangular Nos.
3	1	4	10	20	35	56	84	120	165	220				Pyramidal Nos.
4	1	5	15	35	70	126	210	330	495					Pentahedroidal Nos.
5	1	6	21	56	126	252	462	792						Hexahedroidal Nos.
6	1	7	28	84	210	462	924							Etc.
7	1	8	36	120	330	792								
8	1	9	45	165	496									
9	1	10	55	220										
10	1	11	66											
11	1	12												
C														

TABLE VI.

Coefficients for Calculating Combinations of 3 Variations or General Table of Figurate Numbers.

vertices coinciding, we shall obtain the *arithmetical pyramid* of three dimensions, as shown in Fig. 1. This has the form of a number of cubes piled in a corner of the room. It can of course be indefinitely extended. In it will be found all classes of combinations for $k = 3$, just as in the plane arithmetical triangle are to be found all classes for $k = 2$. Each surface will be an arithmetical of the old sort, all the new classes of three variations being found in the interior cells.

As the numbers ${}_n C_r$ are binomial coefficients in the expansion of $(a+b)^n$, so the numbers representing combinations of three

variations are the trinomial coefficients arising in the expansion of $(a+b+c)^n$. Thus Table IV enables us at once to write out the expansion of $(a+b+c)^{10}$. In like manner the numbers representing combinations of k variations are k -nomial coefficients arising in the expansion of the n th power of a polynomial of k terms.

Let $k = 4$. Each thing may now be an A or a B or a C or a D. In order to develop and represent all possible classes in their proper relations to each other, use will now have to be made of a three-

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	$\times 1$	
1	10	45	120	210	252	210	120	45	10	1		
	$\times 1$	$\times 2$	$\times 3$	$\times 4$	$\times 5$	$\times 6$	$\times 7$	$\times 8$	$\times 9$	$\times 10$		
2	45	120	210	252	210	120	45	10	1			
	$\times 1$	$\times 3$	$\times 6$	$\times 10$	$\times 15$	$\times 21$	$\times 28$	$\times 36$	$\times 45$			
3	120	210	252	210	120	45	10	1				
	$\times 1$	$\times 4$	$\times 10$	$\times 20$	$\times 35$	$\times 56$	$\times 84$	$\times 120$				
4	210	252	210	120	45	10	1					
	$\times 1$	$\times 5$	$\times 15$	$\times 35$	$\times 70$	$\times 126$	$\times 210$					
5	252	210	120	45	10	1						
	$\times 1$	$\times 6$	$\times 21$	$\times 56$	$\times 126$	$\times 252$						
6	210	120	45	10	1							
	$\times 1$	$\times 7$	$\times 28$	$\times 84$	$\times 210$							
7	120	45	10	1								
	$\times 1$	$\times 8$	$\times 36$	$\times 120$								
8	45	10	1									
	$\times 1$	$\times 9$	$\times 45$									
9	10	1										
	$\times 1$	$\times 10$										
10	1											
	$\times 1$											
C												

TABLE VII.

Method of Calculating Combinations for $n = 10$, $k = 3$, from 10th Line of Arithmetical Triangle, and Coefficients of Table VI. Result is Table IV.

Only the 8 numbers enclosed in heavy lines need be calculated, all the others being taken direct or repetitions.

dimensional arrangement. For to every class containing A, B and C, as shown in Table IV, additional classes may now be formed by exchanging the A's successively for D's. The only proper place to put these new classes is to build them out in the third dimension from the original cells of Table IV from which they were developed. The resulting arrangement will be pyramidal in form because of the regularly decreasing value of A from the corner outward to the last diagonal. Such a pyramid, as we read in the fourth line of the arithmetical triangle, will be composed of 4 corners, 6 edges,

4 surfaces and 1 interior space. These will carry respectively the classes where 1, 2, 3 and 4 variations are present. There will be the following groups of such classes:

1 Variation,	A, B, C, D,	= ${}_4C_1$, = 4
2 Variations,	$\begin{cases} AB & BC & CD \\ AC & BD \\ AD \end{cases}$	= ${}_4C_2$, = 6
3 Variations,	ABC ABD ACD BCD	= ${}_4C_3$, = 4
4 Variations,	ABCD	= ${}_4C_4$, = 1

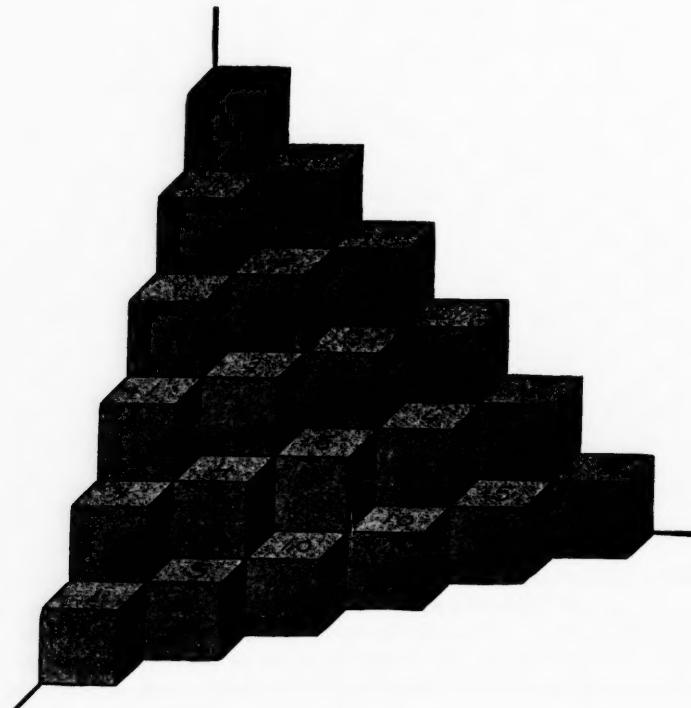


Fig. 1.

Each of the four surfaces of the pyramid will, for $n=10$, be exactly the same as the triangle of Table IV, only as the 12 edges

of the 4 triangles coincide in pairs, thus reducing to 6, and as the 12 corners coincide in threes, thus reducing to 4, we cannot simply repeat the whole triangle four times, but regard must be had for the corners and edges to be omitted. One way would be to represent the corners, edges and interior part of the triangles separately. Another method is shown in Fig. 2. Here the slant sides are supposed to be folded down into the plane of the base, so as to depict all in one plane. The triangle BCD, which is in reality equilateral, is conveniently made right-angled like the others. The sides to be omitted are indicated by dotted lines. A modification of this plan, that economises space, is shown in Fig. 3. A modification of the first method is used in Diag. 1, Table VIII, and in most of the

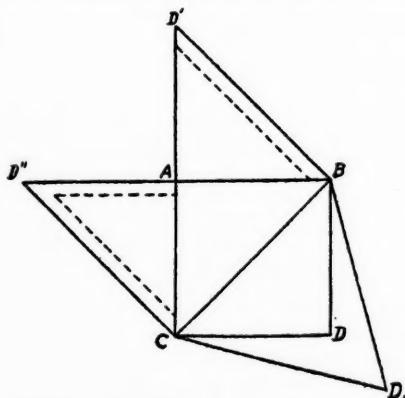


Fig. 2.

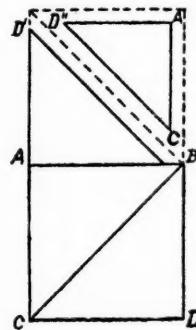


Fig. 3

higher pyramids. The method of Fig. 2 is used in Diag. 5, Table XII, that of Fig. 3 in Diags. 2 and 3 of Table VIII.

The surface of the pyramid being thus represented, and no new numbers to calculate for it, it remains to consider only the representation and calculation of the interior cells, where four variations are present. To do this it will be found most convenient to consider the pyramid as made up of a number of concentric pyramidal shells, each one cell in thickness, like, to use an unsavory simile, the coatings of an onion. Each interior shell will then be exactly similar to the surface shell, which has just been peeled off. It can be represented in the same way, by any one of the three methods described. Analogous to the triangular shells of Table IV,

each cell of the first inner pyramidal shell will contain, beside the numbers indicated of the letters that stand at the vertices of the particular triangle in which it is found, one example of the missing letter. Thus the four triangles of the first inner shell are appropriately lettered, ABC + 1D, ABD + 1C, ACD + 1B, DCB + 1A, as in

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

TABLE VIII, DIAG. 1.
Typical Surface Triangle of Pyramid for
 $n = 10, k = 3$.

The pyramid has 4 corners, 6 edges, 4 surfaces.

Each corner has $1 \times 4 = 4$ or k classes.

" edge " $9 \times 6 = 54$ " $6(n-1)$ "

" surface " $36 \times 4 = 144$ " $4 \frac{(n-1)n}{2}$ " ; or $(n-1)$ th Triangular No.

Total number of classes = 202.

Each corner has $1 \times 4 = 4$ or k Combinations

Each edge " $1,022 \times 6 = 6,132$ or $6(2n-2)$ "

Each inner surface " $55,980 \times 4 = 223,920$ or $4[3^n - 3(2^n - 2) - 3]$ "

Total comb. of surface shell = 230,056.

The four triangles are to be lettered, ABC, ABD, ACD, DCB, as in Figs. 2 and 3.

Diag. 2, Table VIII. Similarly the cells of the second inner shell contain two of the missing letter, and are lettered as in Diag. 3, Table VIII. The third shell, when there is one, will contain 3, the fourth 4, etc., of the missing letter. Each inner shell will have four less cells on a side than the next outer one. Hence until $n = 4$

D		D	5	4	3	2	A
6	2520		7560	12600	12600	7560	2
5	5040	7560		12600	16800	12600	3
4	6300	12600	12600		12600	12600	4
3	5040	12600	16800	12600		7560	5
2	2520	7560	12600	12600	7560		C
A	1	2	3	4	5	6	B
1	720	2520	5040	6300	5040	2520	720
2	2520	7560	12600	12600	7560	2520	2520
3	5040	12600	16800	12600	5040	7560	5040
4	6300	12600	12600	6300	12600	12600	6300
5	5040	7560	5040	12600	16800	12600	5040
6	2520	2520	7560	12600	12600	7560	2520
7	720	2520	5040	6300	5040	2520	720
C	7	6	5	4	3	2	D

TABLE VIII, DIAG. 2.
First Inner Shell of Pyramid.

No. of classes = 74. No. of comb. = 591,720. 7 Nos. to calculate.

D		D		A
3	25200			C
A	2	3	4	B
2	18900	25200	18900	4
3	25200	25200	25200	3
4	18900	25200	18900	2
C	4	3	2	D

TABLE VIII, DIAG. 3.
Second Inner Shell of Pyramid.

No. of classes = 10. No. of comb. = 226,800. 2 Nos. to calculate.

Summary of Pyramid.

4 Classes of 1 Variation having	4 Combinations.
54 " " 2 Variations "	6,132 "
144 " " 3 "	223,920 "
84 " " 4 "	818,520 "
286 " " all "	1,048,576 "
	= 11th Pyramidal Number. = 4 ¹⁰ .

there will be no inner shells. The number of inner shells will always be $n/4$ disregarding the remainder. The first inner shell will be numbered from 1 to $n-3$, the second from 2 to $n-2 \cdot 3$, the third from 3 to $n-3 \cdot 3$; in general, if t is the number of the shell it will be numbered from t to $n-3t$. These inner shells we shall call for short the first tetra, second tetra, etc. They are not really tetrahedral in shape, being composed mostly of right-angled, instead of equilateral triangles, but it will be convenient to call them as designated.

In Table VIII is worked out this case for $n=10$. Diag. 1, which represents the typical surface triangle ABC, is the same as in Table IV. Only the interior portion, enclosed in heavy lines, is exactly repeated on the other three surfaces. The edges, as stated, are repeated six, the corners four times. In reading the triangles it must be remembered that the numbers along the edges refer to the letters that stand at the acute angles. For the inner shells one or two of the missing letter are to be added, and the remainder of the n things are then of the letter that stands at the right angle.

The general formula for calculating the number of combinations corresponding to any interior cell is $n!/A!B!C!D!$. We need only consider in each case the typical triangle ABC+ t D. Since for the first tetra, D=1, the general formula becomes

$$\frac{n!}{B!C!1![n-(B+C+1)]!}.$$
 If we give various values to B and

C, and put them in their proper places in the triangle, we obtain Table IX. It is apparent that this table has all the same regularities as Table V, so that we could here also obtain the interior terms from the edge terms of either columns, lines or diagonals, by determining proper coefficients. But we do not yet know the edge terms. These must themselves be derived somehow. If in imagination we follow any diagonal of the tetra out beyond the latter to where it pierces the surface of the pyramid, we shall find that it ends in a term that is suitable for our calculation. This feat of the imagination may not seem so easy, but the following plan may help.

Suppose Table V composed of a horizontal layer of cubes. Then Table IX, also composed of a horizontal layer of cubes, is to be set down on top of it, so that its first term lies upon the second term,

A	1	2	3	4	5	6	B
1	$1!$	$1!$	$n!$	$n!$	$n!$	$n!$	$6!$
2	$1!$	$1!$	$1!$	$1!$	$1!$	$1!$	$6!$
3	$1!$	$1!$	$1!$	$1!$	$1!$	$1!$	$6!$
4	$1!$	$1!$	$1!$	$1!$	$1!$	$1!$	$6!$
5	$1!$	$1!$	$1!$	$1!$	$1!$	$1!$	$6!$
6	$1!$	$1!$	$1!$	$1!$	$1!$	$1!$	$6!$
C							

TABLE IX.

Formulas for calculating Typical Triangle ABC+1D of First Inner Pyramidal Shell.

General Formula $\frac{B(C+1)[(n-(B+C+1))]}{n!}$

Triangle is numbered from 1 to $n-3$. Table ends when $n=B+C+1$.

second column, of Table V. The latter now twice repeated, but with one line removed, is to be supposed set up on edge so as to enclose the right angle of IX with two vertical walls. Then it

is not difficult to see that the diagonal of the first term of IX pierces these back walls in the third term of the second line. The next diagonal of IX, of course, meets the fourth term, etc. By

A	1	2	3	4	5	6	7	8	B
1	2	3	4	5	6	7	8	9	Natural Nos.
2	3	6	10	15	21	28	36		Triangular Nos.
3	4	10	20	35	56	84			Pyramidal Nos.
4	5	15	35	70	126				Etc.
5	6	21	56	126					
6	7	28	84						
7	8	36							
8	9								
C									

TABLE X.

Coefficients for Calculating Typical Triangle of First Inner Pyramidal Shell.

Table is of Indefinite Extent.

A	1	2	3	4	5	6	7	B
1	$2 \times$ 360	$3 \times$ 840	$4 \times$ 1260	$5 \times$ 1260	$6 \times$ 840	$7 \times$ 360	$8 \times$ 90	
2	$3 \times$ 840	$6 \times$ 1260	$10 \times$ 1260	$15 \times$ 840	$21 \times$ 360	$28 \times$ 90		
3	$4 \times$ 1260	$10 \times$ 1260	$20 \times$ 840	$35 \times$ 360	$56 \times$ 90			
4	$5 \times$ 1260	$15 \times$ 840	$35 \times$ 360	$70 \times$ 90				
5	$6 \times$ 840	$21 \times$ 360	$56 \times$ 90					
6	$7 \times$ 360	$28 \times$ 90						
7	$8 \times$ 90							
C								

TABLE XI.

Method of Calculating Triangle ABC of Diag. 2, Table VII, from Second Line of Surface Triangle and Coefficients of Table X.

comparison of the two tables it is in fact seen that the factor $n!/[n-(B+C+1)]!$, which is constant along the diagonals of IX, is the same in the surface terms in which they end. It is only necessary then to determine the proper coefficients. This can be done as before by factoring the general expression, thus:

$$\frac{n!}{B! C! 1! [n-(B+C+1)]!} = \frac{n!}{(B+C)! 1! [n-(B+C+1)]!} \times \frac{(B+C)!}{B! C!} = "C_B \times C_B C_B$$

Giving B and C their various values as before, we obtain Table X of the coefficients. It is seen at once that this table is exactly the same as Table VI for the surface triangle, except that the first line and column are omitted. The calculation of these inner terms hence becomes extremely simple, and may be reduced to the following rule.

The m th line of the first tetra is derived from the 2d line of the surface triangle by discarding $m+1$ terms, and multiplying the remaining terms by the $(m+1)$ -hedroidal numbers in order, beginning with the second.

A similar investigation will lead to a similarly simple result for the second tetra, which may be reduced to the following rule:

The m th line of the second tetra is derived from the 3d line of the surface triangle, by discarding $m+3$ terms, and multiplying the remaining terms by the $(m+2)$ -hedroidal numbers in order, beginning with the third.

Similar rules may be derived for the succeeding tetra, but if we call t the number of the tetra we may combine them all in one general rule as follows:

The m th line of the t th tetra is derived from the $(t+1)$ th line of the surface triangle by discarding $2t+m-1$ terms and multiplying the remaining terms by the $(t+m)$ -hedroidal numbers in order, beginning with the $(t+1)$ th.

This rule is general not only for all the inner tetras, but by putting t in it equal to 0 it reduces to the rule previously given for the surface triangle, which thus may be considered as the 0-tetra. This one rule hence covers all cases up to the present.

If we construct a series of pyramids, like that of Table VIII, for the successive values of n from 0 up, but give each a thickness of one cell in the direction of the fourth dimension, and pile the successive pyramids so that their A vertices are adjacent to each other in the direction of this dimension, then we shall obtain the four-dimensional arithmetical pyramid. Each three-dimensional pyramid will be a slice of the four-dimensional one, perpendicular to its fourth-dimensional axis, just as each two-dimensional diagram of Fig. 1 is a slice of the three-dimensional pyramid. Each cubical cell will now acquire a thickness equal to its edge in the direction of the fourth dimension and so become a four-dimensional cube, or teseract as it is sometimes called. The whole system will of course contain all classes of combinations up to four variations.

II.

In Part I we have dealt with the combinations of any number of things, each capable of 1, 2, 3 or 4 variations, and found that all possibilities could be represented by tables, having respectively 0, 1, 2 and 3 dimensions, viz., by the point, line, triangle and triangular pyramid. In each case we required a table of $k-1$ dimensions. Hence if we allow more than four variations we must, by the same rule, step out into space of higher dimensions, making use in each case of a $(k-1)$ -dimensional pyramid.

Let us first take the case of $k=5$. Call the variations A, B, C, D and E. By reasoning exactly analogous to that of the case $k=4$, it is clear that from every ABCD cell of the three-dimensional pyramid can be developed a series of new cells equal to the number of A in that cell, by exchanging successively the A's for E's. The only proper place to put these new cells is to build them out from the respective ABCD cells from which they were developed, in the direction of the *fourth* dimension. Because of the regularly diminishing number of the A's in the cells, in passing outward from the A vertex toward the BCD plane, the new solid developed will have the form of a four-dimensional pyramid, analogous to the three-dimensional pyramid previously described. We shall call it a *pentahedroid*, or a *penta* for short, though it is really right-angled instead of equilateral. This penta, as shown by the fifth line of the arithmetical triangle, is bounded by 5 corners, 10 edges, 10 triangular surfaces and five tetrahedra, all enclosing an interior four-dimensional space. These configurations will carry respectively the classes of 1, 2, 3, 4 and 5 variations. The typical triangles of the classes of 1, 2, 3 and 4 variations will be exactly the same as before, except for the different number of repetitions. The five bounding tetras will have interior shells exactly the same as those of diagrams 2 and 3 of Table VIII, and these being independent of one another will be repeated in entirety five times. The 20 surface triangles of the 5 tetras however coincide in pairs, reducing to 10; the 30 edges coincide in threes, reducing to 10; the 20 corners coincide in fours, becoming 5, as already stated. In other words 4 instead of 3 edges now radiate from every vertex, 3 instead of 2 planes from every edge, while every plane divides 2 adjacent tetras from each other.

These 10 surface triangles and the interior shells of the 5

bounding tetras constitute the surface or zero shell of the pentahedroid. The interior space can be considered as before to be made up of concentric pentahedroidal shells, each one cell in thickness in the direction of the fourth dimension. Each such shell will be exactly similar to the surface shell. It will have the same number and kind of boundaries, and can hence be represented in just the same way, viz., by 10 surface triangles, and the interior shells of the 5 bounding tetrahedra. The latter will be called: the first inner tetra shell of the first inner penta shell, second tetra of first penta, etc.

Each inner penta shell will have five less cells on a side than the next outer shell. There will therefore be $n/5$, neglecting remainder, such inner shells. Each will contain one more of each of the two missing letters. The typical triangles, which we shall call the surface triangles of the inner pentas, will be lettered and numbered as follows:

1st penta,	ABC + 1D + 1E	1 to $n-4$
2d " "	ABC + 2D + 2E	2 to $n-2 \cdot 4$
3d " "	ABC + 3D + 3E	3 to $n-3 \cdot 4$
p th " "	ABC + p D + p E	p to $n-4p$

The tetras of the inner pentas will be lettered and numbered as follows:

NAME OF TETRA	LETTERING	NUMBERING	NO. OF INNER TETRA SHELLS
1st tetra of 1st penta	ABC + 2D + 1E	2 to $n-3-4$	$\frac{n-5}{4}$
2d " " 1 "	ABC + 3D + 1E	3 to $n-2 \cdot 3-4$	
t th " " 1 "	ABC + $(t+1)D+1E$	$t+1$ to $n-3t-4$	
1st " " 2d "	ABC + 3D + 2E	3 to $n-3-2 \cdot 4$	$\frac{n-2 \cdot 5}{4}$
2d " " " "	ABC + 4D + 2E	4 to $n-2 \cdot 3-2 \cdot 4$	
t th " " " "	ABC + $(t+2)D+2E$	$t+2$ to $n-3t-2 \cdot 4$	
t th " " p th "	ABC + $(t+p)D+pE$	$t+p$ to $n-3t-4p$	$\frac{n-5p}{4}$

Of course for the other triangles all combinations of the five letters will be taken. This case for $n=10$ is worked out in Table XII. Similar tables can be made for other values of n .

TABLE XII.

Pentahedroidal Pyramid for $n = 10, k = 5$.

(Four Dimensions.)

Boundaries: 5 Corners, 10 Edges, 10 Surfaces, 5 Tetrahedra.

Diag. 1. Surface Penta Shell, Surface Triangles.

Typical Triangle ABC, same as Diag. 1, Table VIII, but

5 Corners \times 1 cell each = 5 Cells.10 Edges \times 9 cells " = 90 "10 Surfaces \times 36 cells " = 360 "

Total Surface Cells of Surface Shell 455 "

Each Corner has $1 \times 5 =$ 5 Combinations." Edge " $1,022 \times 10 =$ 10,220 "" Surface " $55,980 \times 10 =$ 559,800 "

Total of Surface Triangles = 570,025 "

The Ten Triangles are lettered,

ABC	ACD	ADE	BCD	BDE	CDE
ABD	ACE		BCE		
ABE					

Diag. 2. First Inner Tetra Shell of Surface Penta Shell.

Typical Triangle ABC + 1D, same as in Diag. 2, Table VIII.

The Shell contains 5 such Pyramids, hence

 $5 \times 74 = 370$ Cells, and $5 \times 591,720 = 2,958,600$ Combinations.

The Five Pyramids are to be lettered,

ABCD	ABCE	ABDE	ACDE	BCDE
------	------	------	------	------

Each Pyramid is composed of 4 Triangles, making 20 in all for the Shell.

Those of the first Pyramid ABCD are lettered,

ABC + 1D, ABD + 1C, ACD + 1B, BCD + 1A.

Similarly the other 4 Pyramids are lettered.

Diag. 3. Second Inner Tetra Shell of Surface Penta.

Typical Triangle ABC + 2D, same as Diag. 3, Table VIII.

This Diag. five times repeated gives,

 $5 \times 10 = 50$ Cells, and $5 \times 226,800 = 1,134,000$ Combinations.

Lettering same as for first Shell.

Total of Inner Tetra Shells 4,092,600 Comb.

Total of entire Surface Penta Shell, .. 4,662,625 "

TABLE XII (Continued).

A	1	2	3	4	5	6	B
1	5040	15120	25200	25200	15120	5040	
2	15120	87800	50400	37800	15120		
3	25200	50400	50400	2500			
4	25200	37800	25200				
5	15120	15120					
6	5040						
C							

TABLE XII, DIAG. 4.

First Inner Penta Shell.

Typical Surface Triangle
ABC+1D+1E

5 Corners \times 1 Cell each = 5 Cells
 10 Edges " \times 4 Cells " = 40 "
 10 Surfaces \times 6 Cells " = 60 "
 (Shown within the heavy lines)

Total = 105 "

5 Corners with 5,040 Combinations each = 25,200 Comb.
 10 Edges " 80,640 " " = 806,400 "
 10 Surfaces " 264,600 " " = 2,646,000 "
 Total, of Surface Triangles, First Inner Penta = 3,477,600 "

Lettering.

The Ten Triangles of the First Inner Penta Shell are to be lettered thus:

ABC + 1D + 1E	ACD + 1B + 1E	ADE + 1B + 1C
ABD + 1C + 1E	ACE + 1B + 1D	
ABE + 1C + 1D		
BCD + 1A + 1E	BDE + 1A + 1C	CDE + 1A + 1B
BCE + 1A + 1D		

DIAG. 5.

First Inner Tetra Shell of First Inner Penta Shell. Typical Triangle ABC+2D+1E.

Complete Diag. according to Fig. 2 Text.

The 5 Tetra contain $4 \times 5 = 20$ Cells and $75600 \times 4 \times 5 = 1,512,000$ Comb.

No further Tetra Shells to the First Inner Penta.

Triangles are lettered ABC+2D+1E, DCB+2A+1E; ACD+2B+1E, ABD+2C+1E are empty.

Similarly the other Four Tetras.

D		
3		
D	3	A
		2
	2	75600
	3	75600
C	3	2
		D

TABLE XII (Concluded).

A	2	B
2	113400	
C		

DIAG. 6.
Second Inner Penta Shell. Typical Triangle
ABC+2D+2E.

This Shell has but One Cell (since $n/5=2$) and contains 2A, 2B, 2C, 2D, 2E.

Hence Total Combinations of Second Penta Shell = $10!/(2!)^5 = 113,400$.
Total of Inner Pentas (Diags. 4, 5 and 6) = 5,103,000 Comb.

Summary of Pentahedroid.

5 Classes of 1 Variation	having	5 Combinations.
90	" 2 Variations	10,220 "
360	" 3 "	559,800 "
420	" 4 "	4,092,600 "
126	" 5 "	5,103,000 "
1001	" all "	9,765,625 "
		= 11th Pentahedroidal Number.
		= 5^{10} .

The general formula for any interior cell of the penta is $n!/A!B!C!D!E!$. For the surface triangles of the first inner penta this reduces to $\frac{n!}{B!C!1!1![n-(B+C+2)]!}$. Assigning B and C their various values, we obtain Table XIII, for the typical triangle, lettered ABC+1D+1E. Now the typical triangle of the first inner tetra of the surface pyramids is lettered ABC+1D, and its general formula is $\frac{n!}{B!C!1![n-(B+C+1)]!}$. It differs from the present only in the one E lacking. If we compare Table IX with XIII it is seen that the second column of the former coincides with the first column of the latter in the number of A's present in each cell, as shown by the last factor of the denominator. Further it is readily seen that every term of the latter is just double the corresponding term of the former. Comparing further the second column of XIII with the third of IX we find that every term of the former is treble the corresponding term of the latter, and so on for the following columns. We may therefore reduce this case to the following rule:

The surface triangle of the first inner penta shell is derived from the first inner tetra of the surface shell by discarding one column and multiplying the remaining columns successively by the natural numbers in order beginning with the second.

Going through a similar process for the surface triangles of the second penta, we should find that these are derived from the second tetra of the surface shell by discarding 2 columns and multiplying the remaining columns by the triangular numbers in order beginning with the third.

A	1	2	3	4	5	B
1	$\frac{n!}{1!2!1!1!(n-4)}$	$\frac{2!1!1!1!(n-5)}{2!1!1!1!(n-4)}$	$\frac{3!1!1!1!(n-6)}{3!1!1!1!(n-5)}$	$\frac{4!1!1!1!(n-7)}{4!1!1!1!(n-6)}$	$\frac{5!1!1!1!(n-8)}{5!1!1!1!(n-7)}$	
2	$\frac{n!}{1!2!1!1!(n-5)}$	$\frac{2!2!1!1!(n-6)}{2!2!1!1!(n-5)}$	$\frac{3!2!1!1!(n-7)}{3!2!1!1!(n-6)}$	$\frac{4!2!1!1!(n-8)}{4!2!1!1!(n-7)}$		
3	$\frac{n!}{1!3!1!1!(n-6)}$	$\frac{2!3!1!1!(n-7)}{2!3!1!1!(n-6)}$	$\frac{3!3!1!1!(n-8)}{3!3!1!1!(n-7)}$			
4	$\frac{n!}{1!4!1!1!(n-8)}$	$\frac{2!4!1!1!(n-8)}{2!4!1!1!(n-7)}$				
5	$\frac{n!}{1!5!1!1!(n-8)}$					
C						

TABLE XIII.
Formulas for Calculating
First Inner Penta Shell.
Typical Surface Triangle ABC+1D+1E
General Formula $\frac{n!}{B!C!1!1![n-(B+C+2)]!}$
Table Ends when $n=B+C+2$.

Finally in general we should find:

The surface triangle of the p th penta is derived from the p th surface tetra by discarding the first p columns and multiplying

the remaining columns by the $(p+1)$ -hedroidal numbers in order, beginning with the $(p+1)$ th.

Examining similarly the inner tetras of the penta shells we find that the first tetra of the first penta is lettered $ABC + 2D + 1E$, while the second tetra of the surface is lettered $ABC + 2D$, differing again only by the one E lacking. Hence the former may be derived from the latter in a manner similar to the surface triangles of the pentas. Without going through all the details it may at once be stated that the following general rules may easily be derived:

The t th tetra of the *first* inner penta is derived from the $(t+1)$ th surface tetra by discarding the first column and multiplying the remaining columns by the natural numbers in order beginning with the $(t+2)$ d.

The t th tetra of the *second* inner penta is derived from the $(t+2)$ d surface tetra by discarding two columns and multiplying the remaining columns by the triangular numbers in order, beginning with the $(t+3)$ d.

Finally we may set up the following perfectly general rule for any tetra:

The t th tetra of the p th penta is derived from the $(t+p)$ surface tetra by discarding the first p columns and multiplying the remaining columns successively by the $(p+1)$ -hedroidal numbers in order, beginning with the $(t+p+1)$ th.

Substituting in the above $t=0$, we get the rule for the surface triangles of any inner penta, previously given, so that this rule is perfectly general for all inner pentas and their attached tetras.

Finally let $k=6$, viz., A, B, C, D, E and F. This is the case which is presented by a number of dice, each one of which may fall on any one of its six faces. We shall now require for proper representation of all classes a five-dimensional pyramid, or hexahedroid, or hexa as we shall call it for short. From the sixth line of the arithmetical triangle we find that such a figure is bounded by 6 corners, 15 edges, 20 surfaces, 15 tetrahedra, 6 pentahedra, and contains one interior five-dimensional space. These will carry the classes of 1, 2, 3, 4, 5 and 6 variations respectively. The classes of 1, 2 and 3 variations will be represented by the 20 surface triangles, each exactly the same as the previous cases except that regard must be had for the proper number of repetitions of the edges and corners. The classes of 4 variations will be represented by the proper number of surface tetra shells, exactly similar to

NAME OF BOUNDARY	LETTERING	NUMBERING	NO. OF SHELLS	TRIANGLES PER SHELL
Surface Triangles of 1st Hexa	ABC + 1D + 1E + 1F	1 to $n-5$		20
ℓ th Tetra	" " "	$\ell+1$ " $n-3\ell-5$	$\ell = \frac{n-6}{4}$	$15 \times 4 = 60$
Surface of 1st Penta	" "	2 " $n-4-5$		$6 \times 10 = 60$
ℓ th Tetra	" " "	$\ell+2$ " $n-3\ell-4-5$	$\ell = \frac{n-5-6}{4}$	$6 \times 5 \times 4 = 120$
Surface " ρ th "	" " "	$\rho+1$ " $n-4\rho-5$	$\rho = \frac{n-6}{5}$	$6 \times 10 = 60$
ℓ th Tetra	" " "	$\ell+\rho+1$ " $n-3\ell-4\rho-5$	$\ell = \frac{n-5\rho-6}{4}$	$6 \times 5 \times 4 = 120$
Surface Triangles	" " "	h " $n-h$	$h = \frac{n}{6}$	20
ℓ th Tetra	" " "	$\ell+h$ " $n-3\ell-5h$	$\ell = \frac{n-6h}{4}$	$15 \times 4 = 60$
Surface of ρ th Penta	" " "	$\rho+h$ " $n-4\rho-5h$	$\rho = \frac{n-6h}{5}$	$6 \times 10 = 60$
ℓ th Tetra	" " "	$\ell+\rho+h$ " $n-3\ell-4\rho-5h$	$\ell = \frac{n-5\rho-6h}{4}$	$6 \times 5 \times 4 = 120$

TABLE XIV. Lettering and Numbering of a Hexahedroid.

Diagrams 2 and 3 of Table VIII but 15 times repeated for the 15 bounding tetras. The classes of 5 variations will be represented by the proper number of surface penta shells, with their accompanying inner tetra shells, exactly similar to Diagrams 4, 5 and 6 of Table XII, but each 6 times repeated for the 6 bounding pentahedroids. Hence it remains only to consider the cells of the interior five-dimensional space. As before, we shall consider the interior to be made up of concentric inner hexa shells, each one cell in thickness in the direction of the fifth dimension. Each of these inner shells will have the same number and kind of boundaries as the surface or zero hexa just described, and will therefore be represented by the same series of diagrams, viz., 20 surface triangles, with their 15 edges and 6 corners, 15 surface tetra, with their inner shells, 6 bounding pentas, each in turn represented as in Table XII, by 10 surface triangles, with their 10 edges and 5 corners, and by five bounding tetras with their inner shells. Each inner hexa will have six cells less on a side than the next outer one. The number of such inner hexa shells will therefore be $n/6$, neglecting remainder.

The lettering and numbering of the surface triangles, tetras and pentas, with the tetras of the latter, is the same as in the previous case. Hence it only remains to show the numbering and lettering of the inner hexa shells. This is done in Table XIV. The whole table can be developed from the general formulas of the last line by substituting the proper values of t , p and h . In fact by substituting 0 for any of them these formulas will give the surface configurations, and hence the pentahedroid of the previous case. For example, if we put them all equal to zero we get that the surface triangle is lettered ABC and numbered from 0 to n . Also it will have $n/3$ inner triangular shells.

It remains to consider how the combinations for each interior class may be calculated. Without going through the details we may at once state that a perfectly general rule may be set up as follows:

The t th tetra of the p th penta of the h th hexa is derived from the $(t+p)$ th tetra of the h th penta by discarding the first $(p+h)$ columns and multiplying the remaining columns successively by the $(p+h+1)$ -hedroidal numbers in order, beginning with the $(t+p+h+1)$ th.

DIS. No.	VALUES TO ASSIGN	NAME OF SHELL	LETTERING	NUMBERING	DERIVE FROM SHELL	DIS- CARD	MULTIPLY REMAIG BY ($\frac{p+h}{p+h}$) Col. head'd Nos.	BEGIN WITH the $\frac{1+p}{p+h}$ $\frac{n-5p-6h}{p+h}$ $\frac{n}{r}$	NUMBER OF INNER SHELLS
1	0	0	0	6	Surface Tri- angles of 0 Hexa	$ABC + (p+h)D$ $+(p+h)E + hF$	$t+p+h$ $n-3t-4p-5h$		
2	1	0	0	4	1st Tetra of 0 Hexa	$ABC + 1D$	$0 to n$ = 10	10th Line Arith. Triang	$\frac{n}{6} = 1$ Hexa
3	2	0	0	2d	Tetra of 0 Hexa	$ABC + 2D$	$1 " n-1 \cdot 3 = n-3 = 7$	2d Line Sur- face Triangle Terms	$\frac{10}{4} = 2$ Tetra
4	0	1	0	5	1 Penta of 0 Hexa	$ABC + 1D + 1E$	$2 " n-2 \cdot 3 = n-6 = 4$	3d Line Sur- face Triangle Terms	$\frac{4}{4}$ Nat. Trian.
5	1	1	0	4	1st Tetra of 1st Penta of 0 Hexa	$ABC + 2D + 1E$	$1 " n-1 \cdot 4 = n-4 = 6$	1 Col. Natural	$\frac{10}{5} = 2$ Penta
6	0	2	0	2d	Penta of 0 Hexa	$ABC + 2D + 2E$	$2 " n-1 \cdot 3-1 \cdot 4$ $= n-7 = 3$	2d Tetra Col.	$\frac{10-1 \cdot 5}{4} = 1$ Tetra.
7	0	0	1	Surface Trian- gles of 1st Hexa	$ABC + 1D + 1E + 1F$	$2 " n-2 \cdot 4 = n-8 = 2$	2d Tetra Col.	Triangular	3d
8	1	0	1	4	1st Tetra of 1st Hexa	$ABC + 2D + 1E + 1F$	$1 " n-1 \cdot 5 = n-5 = 5$	1st Penta	2d
						$2 " n-1 \cdot 3-1 \cdot 5$ $= n-8 = 2$	1st Tetra of 1st Penta	$\frac{10-1 \cdot 6}{4} = 1$ Tetra	

TABLE XV. Preliminary Scheme, Hexahedroid for $n=10, k=6$.

Derived from General Formulas. (Five Dimensions).

Each Hexa is bounded by 6 Corners, 15 Edges, 20 Surfaces, 15 Tetrahedra, 6 Pentahedroids.
 " Penta " " " 5 " 10 " 10 " 5 "
 " Tetra " " " 4 " 6 " 4 "

TABLE XV (Continued).

Hexahedroid for $n = 10, k = 6$.

Diag. 1. Surface Triangles of Surface or Zero Hexa Shell.

Typical Triangle ABC, same as Diag. 1, Table VIII, but

6 Corners	containing 1 cell each	= 6 Cells.
15 Edges	" 9 cells "	= 135 "
20 Surfaces	" 36 cells "	= 720 "
Total of the 20 Surface Triangles = 861 "		

Each Corner	has $1 \times 6 =$	6 Combinations.
" Edge	" $1,022 \times 15 =$	15,330 "
" Surface	" $55,980 \times 20 =$	1,119,600 "
Total of Surface Triangles = 1,134,936 "		

The Triangles are lettered,

ABC	ACD	ADE	AEF	BCD	BDE	BEF	CDE	CEF	DEF
ABD	ACE	ADF		BCE	BDF		CDF		
ABE	ACF			BCF					
ABF									

Diag. 2. First Inner Tetra Shell of Surface Hexa.

Typical Triangle ABC + 1D, same as Diag. 2, Table VIII, but the Shell is composed of 15 such Pyramids, contains hence

 $15 \times 74 = 1110$ cells, $15 \times 591,720 = 8,875,800$ Combinations.

The Pyramids are lettered,

ABCD	ABDE	ABEF	ACDE	ACEF	ADEF
ABCE	ABDF		ACDF		
ABCF					
			BCDE	BCEF	BDEF
				BCDF	CDEF

Each Pyramid is composed of 4 Triangles, hence 120 in all.

Diag. 3. Second Inner Tetra of Surface Hexa.

Typical Triangle ABC + 2D, same as Diag. 3, Table IX, but 15 times repeated, gives:

 $15 \times 10 = 150$ Cells, $15 \times 226,800 = 3,402,000$ Combinations

Lettering similar to First Tetra.

Sum of the two Tetra Shells = 1260 Cells, with, 12,277,800 Comb.

Diag. 4. Surface Triangles of First Inner Penta Shell of Surface Hexa.

Typical Triangle ABC + 1D + 1E, same as Diag. 4, Table XII, but Shell is composed of 6 such Pentahedroids, hence contains

$6 \times 5 = 30$	Corner Cells
$6 \times 40 = 240$	Edge Cells
$6 \times 60 = 360$	Surface Cells
$6 \times 105 = 630$	Total Cells.

TABLE XV (Continued).

$$\begin{aligned}
 6 \times 25,200 &= 151,200 \text{ Combinations, in Corners} \\
 6 \times 806,400 &= 4,838,400 \text{ Combinations, in Edges} \\
 6 \times 2,646,000 &= 15,876,000 \text{ Combinations, in Surfaces} \\
 6 \times 3,477,600 &= 20,865,600 \text{ Combinations in All.}
 \end{aligned}$$

The 6 Pentas having 10 Triangles each give 60 in all.
The 6 Pentas are lettered,

ABCDE ABCDF ABCEF ABDEF ACDEF BCDEF

The 10 Triangles of the first Penta ABCDE are lettered the same as the 10 Triangles of Diag. 4, Table XII. The remaining Pentas are similarly lettered.

Diag. 5. First Inner Tetra of First Penta Shell.

Typical Triangle ABC + 2D + 1E, same as Diag. 5, Table XII.

The 5 Tetra of this Diag. are repeated 6 times, giving:

$$6 \times 20 = 120 \text{ Cells, } 6 \times 1,512,000 = 9,072,000 \text{ Combinations.}$$

The 6 Pentas having 5 Tetras having 4 Surfaces each give $6 \times 5 \times 4 = 120$ Triangles in all.

The 5 Tetras of the first Penta, ABCDE, will be lettered as in Table XII, the others similarly.

Diag. 6. Surface Triangles of Second Inner Penta Shell.

Typical Triangle ABC + 2D + 2E, same as Diag. 6, Table XII.

This Shell 6 times repeated gives:

$$6 \times 1 \text{ Cell} = 6 \text{ Cells, } 6 \times 113,400 = 680,400 \text{ Combinations.}$$

Sum of the Penta Shells (Diags. 4, 5, 6), give

$$756 \text{ Cells containing } 30,618,000 \text{ Combinations.}$$

A	1	2	3	4	5	B
1	30240	75600	100800	75600	30240	
2	75600	151200	151200	75600		
3	100800	151200	100800			
4	75600	75600				
5	30240					
C						

TABLE XV

DIAG. 7

Surface Triangles of First Inner Hexa Shell. Typical Triangle ABC + 1D + 1E + 1F.

$$6 \text{ Corners of 1 Cell each} = 6 \text{ Cells.}$$

$$15 \text{ Edges} " 3 \text{ Cells each} = 45 "$$

$$20 \text{ Surfaces} " 3 \text{ Cells each} = 60 "$$

$$\text{Total} = 111 "$$

$$6 \text{ Corners have } 30,240 \text{ Comb. each} = 181,440 \text{ Combinations.}$$

$$15 \text{ Edges} " 252,000 " " = 3,780,000 "$$

$$20 \text{ Surfaces} " 453,600 " " = 9,072,000 "$$

$$\text{Total} = 13,033,440 "$$

Lettering same as surface triangles of surface hexa, but one example of each of the three missing letters added.

TABLE XV (Concluded).

DIAG. 8.

A	2	B
2	226800	
C		

First Inner Tetra of First Inner Hexa. Typical Triangle ABC+2D+1E+1F.

One Cell only but 15 Tetra, giving Total of 15 Cells.
 $15 \times 226,800 = 3,402,000$ Total Combinations.

The 15 tetras are lettered same as in Diag. 2. In the single cell composing each tetra are contained two each of the four letters designating the tetra, and one of each of the two missing letters.

$$\begin{aligned} \text{Total Combinations of First Inner Hexa Shell} \\ = \text{Sum of Diags. 7 and 8} = 16,435,440. \end{aligned}$$

Summary of Hexahedroid.

6 Classes of 1 Variation having	6 Combinations
135 " " 2 Variations "	15,330 "
720 " " 3 " "	1,119,600 "
1260 " " 4 " "	12,277,800 "
756 " " 5 " "	30,618,000 "
126 " " 6 " "	16,435,440 "
3003 " " all " "	60,466,176 "

= 11th Hexahedroidal Number. = 6^{10} .

By putting h equal to zero in this rule it reduces to the one already given for the pentahedroid.

In Table XV is worked out from the general formulas a hexahedroid for $n = 10$. First is given a preliminary table showing the number and kind of diagrams needed. The first line of this table repeats the general formulas from which the whole is derived. No really new diagrams are required until we reach the first inner hexa, and only the surface triangles and the first tetra shell of this, the latter containing too only 1 cell, are developed. It might perhaps be more interesting to use a higher value of n so as to develop more of the inner shells, but the numbers increase so rapidly in size that space forbids. For example, if we used $n = 15$ the total of all the combinations would be $6^{15} = 470,184,984,576$ and 16 diagrams would be required.

Let $k = 7$ or higher. We might go on giving k successively higher values, and so develop a septa, an octa, a nona, etc. But the methods would always be the same, and in every case we should end with a general rule that included all of the previous ones. Hence we may at once give the perfectly general rule that will include all the preceding and all the succeeding, viz.:

The typical triangle of the t tetra, of the p penta, of the h

hexa, of the s septa, of the o octa, of the $q(k-1)$ -hedroidal shell, of the $f(k)$ -hedroidal shell.

- (1) *will be lettered* $ABC + (t+p+h+s+o+\dots+f)D$
 $+ (p+h+s+o+\dots+f)E + (h+s+o+\dots+f)F$
 $+ (s+o+\dots+f)G + \dots + f$ times the k th letter.
- (2) *will be numbered* from $t+p+h+s+o+\dots+f$ to
 $n-3t-4p-5h-6s-7o-\dots-(k-1)f$.
- (3) *will be derived from* the typical triangle of the $(t+p)$ tetra of the h penta, of the s hexa, of the o septa, of the $q(k-1)$ -hedroidal shell, by discarding the first $p+h+s+o+\dots+f$ columns and multiplying the remaining columns successively, by the $(1+p+h+s+o+\dots+f)$ -hedroidal numbers in order, beginning with the $(1+t+p+h+s+o+\dots+f)$ th.
- (4) *and will have on each edge*,
 $n-4t-5p-6h-7s-8o-\dots-kf+1$ cells.
- (5) The number of r -hedroidal shells required will be

$$\frac{n-5p-6h-7s-8o-\dots-fk}{r}$$
, where for r is to be substituted the order of the shell required, and the corresponding letter of the shell in the numerator is then to be omitted.

To apply these rules simply give to the letters t, p, h, s , etc., successively the values, 0, 1, 2, 3, etc., in all combinations, until negative values occur, or until the proper number of shells have been developed. As far as lettering and numbering are concerned, these rules apply to all cases. For derivation they apply only to the inner shells after the surface tetra. The latter and the surface triangles must be calculated line by line, according to the general rule given on page 21. By considering the surface triangles and tetra to be made up of triangular shells, and considering a typical edge of such shells, calling the outer edge the zero shell, a perfectly general rule could be given for all cases. But it would be cumbrous, so that practically we find it better to divide the derivation, as has been done, into two rules.

One may well question whether all the foregoing is very important or useful. Certainly it is not of very great advantage until high values of n and k are reached. Still even in fairly simple cases it is of some help. To show this, Table XVI has been given for $n=4, k=6$. This shows all the ways in which four dice may be

thrown. Here we reach only the *first* surface tetra, and even this has only one cell, viz., the case where all four of the dice show a different number. The number of ways in which this can occur is given directly by $n! = 24$. All the other classes are shown in the surface triangles. There are 6 where one number only appear, 45 where two appear, 60 where three different numbers appear, and 15 where all four dice show different numbers. All the calculations can be made mentally, for when in the surface triangle we have said $6 \times 2 = 12$, we have obtained all of the different numbers. The method of representation enables all the classes to be enumerated without difficulty or doubt, and gives all the detailed information that can be desired. The total number of classes is 126, or the fifth hexahedroidal number. The total of all combinations is $1296 = 6^4$.

I	0	1	2	3	4	II
0	1	4	6	4	1	
1	4	12	12	4		
2	6	12	6			
3	4	4				
4	1					
III						

TABLE XVI.

Hexahedroid for $n=4$, $k=6$.

Diag. 1. Surface Triangles.

Typical Triangle I, II, III.

6	Corners $\times 1$	=	6	Comb. 6×1	=	6	Classes	
15	Edges $\times 14$	=	210	"	15×3	=	45	"
20	Surfaces $\times 36$	=	720	"	20×3	=	60	"
	Total	=	936	"		=	111	"

Diag. 2. First Surface Tetra Shell.

Typical Triangle I, II, III, + 1 IV.

I	1	II
1	24	
III		

15	Tetras $\times 24$	=	360	Comb. 15×1	=	15	Classes
Total of All	= 1296	"		= 126	"		
	= 6^4			= 5th Hexahedroidal No.			

In fact, it was the inquiry of a friend with regard to dice that started the whole investigation.

But whether one concedes to this system any measure of usefulness or not, it affords a striking example of the wonderful inter-relations between numbers and geometry, and adds another to the many remarkable properties of the arithmetical triangle. Pascal, in the work referred to at the beginning, exclaimed: "C'est une chose étrange combien il est fertile en propriétés! Chacun peut s'y exercer." We have exercised ourselves there, and hope that this new property or extension may not prove wholly uninteresting.

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GENERAL RULE FOR CONSTRUCTING ORNATE MAGIC
SQUARES OF ORDERS $\equiv 0 \pmod{4}$.

Take a square lattice of order $4m$ and draw heavy lines at every fourth vertical bar and also at every fourth horizontal bar, thus dividing the lattice into m^2 subsquares of order 4. The "period" consists of the $4m$ natural numbers 1, 2, 3, ..., $4m$. Choose from these *any* two pairs of complementary numbers, that is, pairs whose sum is $4m+1$ and arrange these four numbers, four times repeated, as in a Jaina square (first type) in the left-hand square of the top row of subsquares in the large lattice. It is essential that the Jaina pattern shall contain only one complementary couplet in each of

a_1	b_1	a_1	b_2
a_2	b_2	a_2	b_1
a_1	b_1	a_2	b_2
a_2	b_2	a_1	b_1

Fig. 1.

10	51	15	54	12	49	13	56
23	46	18	43	21	48	20	41
50	11	55	14	52	9	53	16
47	22	42	19	45	24	44	17
26	35	31	38	28	33	29	40
7	62	2	59	5	64	4	57
34	27	39	30	36	25	37	32
63	6	58	3	61	8	60	1

Fig. 2.

its four columns, i. e., if the two pairs are $a_1 a_2$ and $b_1 b_2$, every column must consist entirely of a 's, or entirely of b 's. The first Jaina type can be obtained by using the paths (1, 2), (2, 1) and the order $a_1 b_1 a_2 b_2$ four times repeated. This gives the square shown in Fig. 1, which fulfils the conditions. Proceed in the same way with each of the m subsquares in the top row, *using a different pair of complementaries in each subsquare*. Since the period 1, 2, 3, ..., $4m$ contains $2m$ complementary pairs and two pairs are used for each subsquare, it follows that when the top row of subsquares is filled up, all the $4m$ numbers will have been used.

Now fill all the remaining rows of subsquares in the large lattice with replicas of the top row. The *outline* so constructed can

always be turned over either of its central diagonals without repetition. The resulting square will therefore contain the first $(4m)^2$ numbers without repetition or omission, and it will always have the following magic properties.

A. *The Great Square.....*

1. is magic on its $4m$ rows and $4m$ columns ;
 2. is pandiagonal, i. e., magic on its $8m$ diagonals ;
 3. has Franklin's property of *bent diagonals* in an extended sense; i. e., we can start at any cell in the top row, and proceeding downward bend the diagonal at *any* heavy horizontal bar. It matters not how many times we bend, or at which of the heavy bars, providing only that when the traverse is completed, the number of cells passed over in the one direction (downward to the right) shall be exactly equal to the number passed over in the other direc-

2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1

Fig. 3.

8	48	8	48	8	48	8	48
16	40	16	40	16	40	16	40
48	8	48	8	48	8	48	8
40	16	40	16	40	16	40	16
24	32	24	32	24	32	24	32
0	56	0	56	0	56	0	56
32	24	32	24	32	24	32	24
56	0	56	0	56	0	56	0

Fig. 4.

tion (downward to the left). Similarly we may start at any cell in the left-hand column and, proceeding diagonally to the right, bend the diagonal at *any* heavy vertical bar under the same limitations.

It will be noticed that when the order of the square is $\equiv 4$ (mod. 8), i. e., when m is odd, the central bars are not *heavy bars*, and also the number of rows of subsquares is odd. We cannot therefore in these cases get a magic bent diagonal traverse from top to bottom of the square, but we may stop at the last heavy bar before reaching the bottom of the square, when we shall have a sum $4(m-1)$ times the mean, or we may carry the diagonal beyond the bottom of the square and traverse the top row of subsquares a

27	46	111	106	3	58	135	94	63	22	75	130
112	105	28	45	136	93	4	57	76	129	64	21
34	39	118	99	10	51	142	87	70	16	82	123
117	100	33	40	141	88	9	52	81	124	69	16
25	48	109	108	1	60	133	96	61	24	73	132
113	104	29	44	137	92	5	56	77	128	65	20
36	37	120	97	12	49	144	85	72	13	84	121
116	101	32	41	140	89	8	53	80	125	68	17
30	43	114	103	6	55	138	91	66	19	78	127
110	107	26	47	134	95	2	59	74	131	62	23
31	42	115	102	7	54	139	90	67	18	79	126
119	98	35	38	143	86	11	50	83	122	71	14

Fig. 5. S = 870

115	110	131	158	3	78	245	190	51	94	195	174	19	46	227	222
130	153	114	111	242	191	2	79	194	175	50	95	226	223	18	47
126	99	142	147	14	67	254	179	62	83	206	163	30	35	238	211
143	146	127	98	255	178	15	66	207	162	63	82	239	210	31	34
118	107	134	155	6	75	246	187	54	91	198	171	22	43	230	219
132	157	116	109	244	189	4	77	196	173	52	93	228	221	20	45
123	102	139	150	11	70	251	182	59	86	203	166	27	38	235	214
141	148	125	100	253	180	13	68	205	164	61	84	237	212	29	36
117	108	133	156	5	76	245	188	53	92	197	172	21	44	229	220
129	160	113	112	241	192	1	80	193	176	49	96	225	224	17	48
124	101	140	149	12	69	252	181	60	85	204	165	28	37	236	213
144	145	128	97	256	177	16	65	208	161	64	81	240	209	32	33
119	106	135	154	7	74	247	186	55	90	199	170	23	42	231	210
136	153	120	105	248	185	8	73	200	169	56	89	232	217	24	41
122	103	138	151	10	71	250	183	58	87	202	167	26	39	234	215
137	152	121	104	243	184	9	72	201	168	57	88	233	216	25	40

Fig. 6. S = 2056

second time, when the sum will be $4(m+1)$ times the mean. We can get in these cases a diagonal traverse $4m$ times the mean by inserting at any point one vertical series of four cells between any two heavy bars and then continuing diagonally.

4. The great square is 4-ply, and therefore 4-symmetrical, i. e., we may choose any vertical and any horizontal bar (not

1	382	20	398	3	334	18	397	5	386	16	395	7	388	14	383	9	390	12	391
40	379	21	362	38	377	23	364	36	375	25	366	34	373	27	368	32	371	29	370
381	2	400	19	383	4	398	17	385	6	396	15	387	8	389	13	389	10	392	11
380	39	367	22	378	37	363	24	376	35	365	26	374	33	367	28	372	31	369	30
41	342	60	353	43	344	58	357	45	346	56	355	47	348	54	353	49	350	52	351
80	339	61	322	78	337	63	324	76	335	65	326	74	333	67	328	72	331	69	330
341	42	360	59	343	44	358	57	345	46	356	55	347	48	357	53	340	50	352	51
340	79	321	62	338	77	323	64	336	75	326	66	334	73	327	68	332	71	329	70
31	302	100	319	83	304	98	317	85	306	96	315	87	308	94	313	89	400	92	311
120	299	101	282	118	297	103	286	116	295	105	296	114	293	107	288	112	291	109	290
301	82	320	99	303	84	318	97	305	86	316	95	307	88	314	93	309	90	312	91
300	119	281	102	298	117	283	104	296	115	285	106	294	113	287	108	292	111	289	110
121	266	140	279	133	264	138	277	125	266	136	275	127	268	134	273	129	270	132	271
160	259	141	242	158	257	143	244	156	255	145	246	154	253	147	241	152	251	149	250
267	122	280	139	263	124	278	137	265	126	276	135	267	129	274	133	269	130	272	131
260	169	241	148	258	157	243	144	256	153	245	146	254	153	247	148	262	151	269	150
161	222	180	239	163	224	178	237	165	226	176	235	167	228	174	233	169	230	172	231
200	219	181	202	198	217	183	204	196	215	185	206	194	213	187	208	192	211	189	210
221	162	240	179	228	164	238	177	225	166	236	175	227	167	234	173	229	170	232	171
220	159	201	182	218	197	203	184	216	195	205	186	214	193	207	188	212	191	209	180

Fig. 7.

necessarily heavy bars) and we shall find that any four cells, symmetrically situated with regard to these two bars as axes, will contain numbers whose sum is four times the mean. It follows that any $4m$ cells which form a symmetrical figure with regard to any such axes will contain numbers whose sum is the magic sum of the great square.

1	552	553	48	49	504	505	96	97	456	457	142	145	408	409	192	193	460	461	240	241	342	343	244
575	26	23	530	527	74	71	482	479	122	119	434	431	170	167	486	483	217	215	335	335	266	263	240
24	529	576	25	72	481	578	73	120	433	480	121	162	485	492	169	216	337	337	217	217	249	249	262
534	47	2	551	506	35	50	503	458	143	98	455	410	191	146	407	62	239	194	359	344	242	341	245
9	530	535	46	57	502	507	94	39	452	459	142	147	406	411	190	195	359	363	238	243	310	315	246
573	28	21	532	525	76	69	484	477	124	117	436	429	172	165	486	481	220	213	340	343	263	261	242
22	574	27	70	483	526	75	118	435	478	123	165	387	430	171	214	339	382	219	214	391	344	267	267
535	45	4	549	508	33	52	501	460	141	100	453	412	189	148	405	364	237	190	367	316	265	244	309
5	548	557	44	53	500	329	92	101	452	463	140	149	404	413	188	197	356	363	236	245	308	311	244
571	30	19	534	523	78	67	486	475	126	115	438	427	174	163	390	379	222	211	342	331	270	269	244
20	533	572	29	68	485	324	77	116	437	476	125	164	389	428	173	212	341	380	221	202	293	332	269
538	43	6	547	510	91	54	499	462	139	102	451	414	187	150	403	366	235	198	355	311	283	246	307
7	540	539	42	55	498	511	90	103	430	463	138	151	402	415	186	199	354	367	234	247	306	319	242
569	32	17	536	521	80	65	488	473	128	113	440	425	176	161	392	377	224	209	342	331	270	269	244
18	535	370	31	66	487	322	73	114	439	474	127	162	391	426	173	210	343	382	221	202	293	332	269
560	41	8	545	512	83	58	497	464	137	104	449	416	185	152	401	368	233	200	353	320	281	248	303
9	544	561	40	57	486	513	88	105	448	465	136	153	400	417	184	201	352	369	232	249	304	321	240
567	34	15	538	519	82	63	480	471	130	111	442	423	178	159	384	375	226	207	347	324	265	259	241
16	537	586	33	64	489	340	87	112	441	472	129	160	393	424	174	208	345	376	225	205	297	334	273
562	39	10	543	514	87	58	493	466	135	105	447	418	183	154	389	370	231	202	357	322	279	257	303
11	562	563	38	59	494	515	86	107	446	467	134	155	398	419	182	203	350	371	230	207	342	323	279
565	36	13	560	517	84	61	492	463	132	109	444	421	180	157	396	373	228	205	348	325	270	253	300
14	538	567	45	62	491	478	83	110	443	470	131	158	385	422	178	200	347	374	227	204	299	336	275
564	37	13	571	516	85	66	493	467	133	108	445	420	181	158	397	372	229	204	349	324	277	252	301

Fig. 8.

B. *The Subsquares.....*

5. are balanced Jaina squares, i. e., each of them has the 36 summations of a Jaina and in each case the magic sum is four times the mean number of the great square.

6. They have the property of subsidiary minors, i. e., if we erase any p rows of subsquares, and any p columns of the same and draw the remaining rows and columns together, we have a square with *all* the properties of the original great square.

EXAMPLES

In every case the Jaina pattern quoted above is used. Fig. 2 is an example of order 8 and the complementaries have been paired thus: 2,7 with 3,6; and 4,5 with 1,8. The La Hirean primaries of Fig. 2 are shown in Figs. 3 and 4.

* * *

Fig. 5 is an example of an order 12 square in which the pairing

1. 16 — 2. 15				13. 4 — 14. 3				12. 5 — 11. 6				8. 3 — 7. 10			
1	2	16	15	13	14	4	3	12	11	5	6	8	7	9	10
16	15	1	2	4	3	13	14	5	6	12	11	9	10	8	7
1	2	16	15	13	14	4	3	12	11	5	6	8	7	9	10
16	15	1	2	4	3	13	14	5	6	12	11	9	10	8	7

Fig. 9.

of the complementaries is 3,10 with 4,9; 1,12 with 5,8; and 6,7 with 2,11.

* * *

A square of order 16 is shown in Fig. 6. The couplets in this square are taken thus:

8 and 9 with 7 and 10; 1 and 16 with 5 and 12;

4 and 13 with 6 and 11; 2 and 15 with 3 and 14.

Figs. 7 and 8 show respectively squares of orders 20 and 24 in which the couplets are taken in numerical order, i. e., for order 20, 1 and 20 with 2 and 19; 3 and 18 with 4 and 17, etc.

In Fig. 8 there are 1008 magic diagonal summations. Since we can bend at any heavy bar, the number of bent diagonals from top to bottom, starting at a given cell in the top row, is the same as the

number of combinations of 6 things 3 at a time, viz., 20. Therefore there are $20 \times 24 = 480$ bent diagonals from top to bottom and 480 more from side to side. Adding the 48 continuous diagonals we get 1008.

In the foregoing pages the question of magic knight paths has not been considered. It is, however, easy for all orders > 8 and $\equiv 0$ (mod. 8) to add the knight nasik property without sacrificing any of

1	32	241	240	193	284	49	48	177	176	65	96	113	112	129	160
242	239	2	31	50	47	194	223	66	95	178	175	130	153	114	111
16	17	256	225	208	209	64	33	192	161	80	81	128	97	144	145
255	226	15	18	63	34	207	210	79	82	191	162	143	146	127	98
13	20	253	228	205	212	61	36	189	164	77	84	125	100	141	141
254	227	14	19	62	35	206	211	78	83	190	163	142	147	126	99
4	29	244	237	196	221	52	45	180	173	68	93	116	109	132	157
243	238	3	30	51	46	195	222	67	94	179	174	131	158	115	110
12	21	252	229	204	213	60	37	188	165	76	85	124	101	140	149
251	230	11	22	59	38	203	214	75	86	187	166	139	150	123	102
5	28	245	236	197	220	53	44	181	172	69	92	117	108	133	158
246	235	6	27	54	43	198	219	70	91	182	171	136	155	118	107
8	25	248	233	200	217	56	41	184	163	72	89	120	105	136	153
247	234	7	26	55	42	199	218	71	90	183	170	135	154	119	106
9	24	249	232	201	216	57	40	185	168	73	88	121	104	137	152
250	231	10	23	58	39	202	215	74	87	186	167	138	157	122	103

Fig. 10.

the other features, by a proper choice of the complementary couplets for the subsquare outlines. The example shown in Fig. 9 will explain. It shows the top row of subsquares in a scheme for order 16. The numbers above the squares indicate the couplets used, the Jaina pattern, Fig. 1, being used throughout. The rule is simple: the leading numbers, 1, 13, 12, 8 must sum four times the mean of the period, i. e., 34, while of course no one of them may be a complement of any other. Their complementaries 16,

4, 5, 9, will then have the same sum, and the second members in each square will be similarly related. The square is completed by filling the remaining rows with replicas and turning over a central diagonal. Fig. 10 is a square of order 16 constructed from the outline Fig. 9. It has all the properties of the 16² shown in Fig. 6, and is also magic on its 64 knight paths.

The following is an arrangement of the couplets for a square of order 24:

1.24-4.21	8.17-5.20	10.15-13.12	11.14-16.9	22.3-18.7	23.2-19.6
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C. PLANCK.

HAYWARD'S HEATH, ENGLAND.

ORNATE MAGIC SQUARES OF COMPOSITE ODD ORDERS.

When we consider these orders in the light of the general rule used for orders $\equiv 0 \pmod{4}$ it appears at first sight that they cannot be made to fulfil all the conditions; but it is not essential to the *ply* property, nor to the balanced magic subsquares that the numbers be taken in complementary pairs for the subsquares of the outline. All that is necessary is that the groups of numbers chosen shall all have the same sum.

Suppose, as an illustration, we are dealing with order 15. If we can arrange the first 15 natural numbers in five balanced

4

2	7	15
7	15	2
15	2	7

Fig. 1.

2	6	12	11	9
15	13	8	3	1
7	5	4	10	14

Fig. 2.

columns, three in a column, and form five magic outlines of order 3, using a different column thrice repeated for each outline, we shall have five balanced magic outlines like Fig. 1. These can be arranged in the first row of subsquares with replicas in the following rows. If we can turn this outline upon itself in some way to avoid repetitions, we shall have a magic square which will be 9-ply and with magic subsquares. But will it be pandiagonal?

In the small outlines of 9 cells made from Fig. 1 as a pattern, it

will be noticed that like numbers must always occur in parallel diagonals; therefore if we arrange the five small squares so that like numbers always lie along \diagup diagonals, the great outline will be "boxed" and therefore magic in \diagdown diagonals, but in the \diagup diagonals we shall have in every case only five different numbers each occurring thrice. The problem is thus reduced to finding a magic rectangle 3×5 . We therefore construct such a rectangle by the method of "Complementary Differences"¹ as shown in Fig. 2.

In Fig. 3 we have the five magic outlines constructed from the five columns of the rectangle, and placed side by side with like

2	7	15	6	5	13	12	4	8	11	10	3	9	14	1
7	15	2	5	13	6	4	8	12	10	3	11	14	1	9
15	2	7	13	6	5	8	12	4	3	11	10	1	9	14

Fig. 3.

2	12	9	6	11	15	8	1	13	3	7	4	14	5	10
9	6	11	2	12	1	13	3	15	8	14	5	10	7	4
11	2	12	9	6	3	15	8	1	13	10	7	4	14	5
12	9	6	11	2	8	1	13	3	15	4	14	5	10	7
6	11	2	12	9	13	3	15	8	1	5	10	7	4	14

Fig. 4.

numbers always in the \diagup diagonals, and so disposed that *the number* in any \diagup diagonal is always succeeded (when the diagonal passes across into a neighboring square) by the number which succeeds it in its row in the rectangle.

If an associated square is required the magic rectangle must be associated and the large rectangle of subsquares must also be associated as a whole. It will be noticed that all these conditions will be fulfilled in practice if we write the successive columns of the magic rectangle Fig. 2 along the \diagdown central diagonals of the successive square outlines in the larger rectangle Fig. 3, and fill in all the \diagup diagonals with replicas. If now all the remaining rows of

¹ See "The Construction of Magic Squares and Rectangles by the Method of Complementary Differences," by W. S. Andrews, *Monist*, July, 1910, Vol. XX, No. 3.

subsqares be filled with replicas of the top row it will be found that the whole outline *cannot* be turned over either of its central diagonals without repetitions in the magic, but it *can* be turned successfully *in its own plane*, about its central point through one right angle, without repetitions. (This will bring the top row in coincidence with the left-hand column, so that the right-hand square in Fig. 3 is turned on its side and lies over the left-hand square.) The resulting magic is shown in Fig. 6. It is magic on its 15 rows,

155	28	171	125	88	156	20	178	126	80	163	21	170	133	81
44	211	114	14	181	39	224	106	9	194	31	219	119	1	189
139	98	57	199	68	147	94	53	207	64	143	102	49	203	72
157	30	167	127	90	152	22	180	122	82	165	17	172	135	77
40	213	116	10	183	41	220	108	11	190	33	221	115	3	191
190	103	51	200	73	141	95	58	201	65	148	96	50	208	66
164	16	174	134	76	159	29	166	129	89	151	24	179	121	84
34	218	117	4	188	42	214	113	12	184	38	222	109	8	192
142	105	47	202	75	137	97	60	197	67	150	92	52	210	62
160	18	176	130	78	161	25	168	131	85	153	26	175	123	86
35	223	111	5	193	36	215	118	6	185	43	216	110	13	186
149	91	54	209	61	144	104	46	204	74	136	99	59	196	69
154	23	177	124	83	162	19	173	132	79	158	27	169	128	87
37	225	107	7	195	32	217	120	2	187	45	212	112	15	182
145	93	56	205	63	146	100	48	206	70	138	101	55	198	71

Fig. 5.

S = 1695

15 columns, 30 diagonals and 60 knight paths, also 9-ply and associated. The 25 subsquares of order 3 all sum 339 on their 3 rows and 3 columns. (It is easy to see that only one of them can have magic central diagonals, for a magic of order 3 can only have this property when it is associated, and in this case the mean number must occupy the central cell, but there is here only one mean number, viz., 113, therefore only the central subsquare can have magic diagonals.)

In exactly the same manner as above described, by using the

long rows of the magic rectangle Fig. 2, instead of the short columns, we can construct another ornate magic of order 15.

Fig. 4 shows the first row of 25-celled subsquares constructed from the rows of the rectangle, and using a magic square of order 5 as pattern. If we fill the two remaining rows of subsquares with replicas the outline can be turned over either of its central diagonals. The resulting square is shown in Fig. 7. It is magic on 15 rows, 15 columns, 30 diagonals and 60 knight paths, also 25-ply and asso-

2	127	210	6	125	208	12	124	203	11	130	198	9	134	195
202	15	122	200	13	126	199	8	132	205	3	131	209	1	129
135	197	7	133	201	5	128	207	4	123	206	10	121	204	14
32	157	150	36	155	148	42	154	143	41	160	138	39	164	136
142	45	152	140	43	156	139	38	162	145	33	167	149	31	153
165	137	37	163	141	35	158	147	34	153	146	40	151	144	44
107	172	60	111	170	58	117	169	53	116	175	48	114	179	46
32	120	167	50	118	171	49	113	177	55	108	176	59	106	174
180	47	112	178	51	110	173	57	109	168	56	115	166	54	119
182	82	75	186	60	73	192	79	68	191	85	63	189	89	61
67	195	77	65	193	81	64	188	87	70	183	86	74	181	84
90	62	187	88	66	185	83	72	184	78	71	190	76	69	134
212	22	105	216	20	103	222	19	98	221	25	93	219	29	91
37	225	17	95	223	21	94	218	27	100	213	26	104	211	24
30	92	217	28	96	215	23	102	214	18	101	220	16	99	224

Fig. 6

S = 1695

ciated. Also the nine subsquares of order 5 are balanced nasiks, summing 565 on their 5 rows, 5 columns and 10 diagonals.

The above method can of course be used when the order is the square of an odd number, e. g., orders 9, 25, etc. These have previously been dealt with by a simpler method which is not applicable when the order is the product of different odd numbers.

A similar distinction arises in the case of orders $\equiv 0 \pmod{4}$ previously considered. These were first constructed by a rule which applied only to orders of form 2^m , e. g., 4, 8, 16, 32, etc., but the general rule is effective in every case.

There are two other ornate squares of order 15, shown in Figs. 5 and 8, these four forms of ornate squares being numbered in ascending order of difficulty in construction. Fig. 5 is constructed by using the paths $\begin{Bmatrix} 3, 5 \\ 5, 3 \end{Bmatrix}$ and taking the period from the *continuous diagonal* of the magic rectangle Fig. 2.

Fig. 5 is magic on 15 rows, 15 columns, 30 diagonals, 60 knight paths, and is 9-ply, 25-ply and associated.

The square shown in Fig. 8 has been only recently obtained;

17	132	153	171	86	30	128	151	178	78	22	124	164	170	85
174	81	26	122	162	166	88	18	135	158	179	80	25	127	154
131	152	177	84	21	123	165	173	76	28	130	157	169	89	20
87	24	126	161	167	83	16	133	153	180	79	29	125	160	172
156	176	77	27	129	163	168	90	23	121	155	175	82	19	134
212	12	39	111	191	225	8	31	118	183	217	4	44	110	130
114	186	221	2	42	106	193	213	15	38	119	185	220	7	34
11	32	117	189	216	3	45	113	181	223	10	37	109	194	215
192	219	6	41	107	188	211	13	33	120	184	204	5	40	112
36	116	182	222	9	43	108	195	218	1	35	115	187	214	14
92	207	144	51	71	105	203	136	58	63	97	199	149	50	70
54	66	101	197	147	46	73	93	210	143	59	65	100	202	139
206	137	57	69	96	198	150	53	61	103	205	142	49	74	95
72	99	201	146	47	68	91	208	138	60	64	104	200	145	52
141	56	62	102	204	148	48	75	98	196	140	55	67	94	209

Fig. 7.

S = 1695

for many years the conditions therein fulfilled were believed to be impossible. It is magic on 15 rows, 15 columns and 30 diagonals, and is 3×5 rectangular ply, i. e., any rectangle 3×5 with long axis horizontal contains numbers whose sum is the magic sum of the square. Also the 15 subrectangles are balanced magics, summing 565 in their three long rows and 339 in their five short columns. This square is not associated, and only half of its knight paths are magic.

The three squares of order 15, shown in Figs. 5, 6 and 7, are

described as magic on their 60 knight paths, but actually they are higher nasiks of Class II, as defined at the end of my pamphlet on *The Theory of Path Nasiks*.² Further, the squares in Figs. 6 and 7 have the following additional properties.

Referring to the square in Fig. 7 showing subsquares of order 5; if we superpose the diagonals of these subsquares in the manner described in my paper on "Fourfold Magics" (*The Monist*, Vol. XX, p. 618, last paragraph), we obtain three magic parallelopipeds

37	93	191	81	163	32	99	185	89	160	45	102	188	79	151
167	219	5	59	115	180	222	8	49	106	172	213	11	51	118
135	27	143	199	61	127	18	146	201	73	122	24	140	209	70
97	183	86	156	43	92	189	80	164	40	105	192	83	154	31
212	9	50	119	175	225	12	53	109	166	217	3	56	111	178
30	147	203	64	121	22	138	206	66	133	17	144	200	74	130
187	78	161	36	103	182	84	155	44	100	195	87	158	34	31
2	54	110	179	220	15	57	113	189	211	7	48	110	171	223
150	207	68	124	16	142	198	71	186	28	137	204	65	134	25
82	153	41	96	193	77	159	35	104	190	90	162	38	94	181
47	114	170	224	10	60	117	173	214	1	52	108	170	216	13
210	72	128	19	136	202	63	131	21	148	197	69	125	29	145
157	33	101	186	88	152	39	95	194	85	165	42	98	184	76
107	174	215	14	55	120	177	218	4	46	112	168	221	6	58
75	132	23	139	195	67	123	26	141	208	62	129	20	149	203

Fig. 8.

S = 1695

$5 \times 5 \times 3$. Denoting each subsquare by the number in its central cell, the three parallelopipeds will be:

- I. 53, 169, 117.
- II. 177, 113, 49.
- III. 109, 57, 173.

These three together form an octahedroid $5 \times 5 \times 3 \times 3$ which is associated and magic in each of the four directions parallel to its edges.

If we deal in like manner with Fig. 6 which has subsquares of

order 3 we obtain five magic parallelopipeds of order $3 \times 3 \times 5$ together forming an associated magic octahedroid of order $3 \times 3 \times 5 \times 5$. Since the lengths of the edges are the same as those of the octahedroid formed from Fig. 7 square, these two four-dimensional figures are identical but the distribution of the numbers in their cells is not the same. They can however be made completely identical both in form and distribution of numbers by a slight change in our method of dealing with the square Fig. 6, i. e., by taking the square plates to form the parallelopipeds from the knight paths instead of the diagonals. Using the path $-1, 2$ we get 225, 106, 3, 188, 43 for the first plates of each parallelopiped, and then using $2, -1$ for the successive plates of each, we obtain the parallelopipeds:

I.	225,	8,	31,	118,	183
II.	106,	193,	213,	15,	38
III.	3,	45,	113,	181,	223
IV.	188,	211,	13,	33,	120
V.	43,	108,	195,	218,	1

This octahedroid is completely identical with that previously obtained from Fig. 7, as can be easily verified by taking any number at random and writing down the four series of numbers through its containing cell parallel to the edges, first in one octahedroid and then in the other. The sets so obtained will be found identical.

HAYWARD'S HEATH, ENGLAND.

C. PLANCK.

PANDIAGONAL-CONCENTRIC MAGIC SQUARES OF ORDERS $4m$.

These squares are composed of a central pandiagonal square surrounded by one or more bands of numbers, each band, together with its enclosed numbers, forming a pandiagonal magic square.

The squares described here are of orders $4m$ and the bands or borders are composed of double strings of numbers. The central square and bands are constructed simultaneously instead of by the usual method of first forming the nucleus square and arranging the bands successively around it.

² *The Theory of Path Nasiks*, by C. Planck, M.A., M.R.C.S., printed by A. J. Lawrence, Rugby, Eng.

A square of the 8th order is shown in Fig. 1, both the central 4^2 and 8^2 being pandiagonal. It is 4^2 ply, i. e., any square group of 16 numbers gives a constant total of $8(n^2+1)$, where n = the number of cells on the edge of the magic. It is also magic in all of its Franklin diagonals; i. e., each diagonal string of numbers bending at right angles on either of the horizontal or vertical center lines of the square, as is shown by dotted lines, gives constant totals. In any size concentric square of the type here described, all of its concentric squares of orders $8m$ will be found to possess the Franklin bent diagonals.

The analysis of these pandiagonal-concentric squares is best illustrated by their La Hirean method of construction, which is

45	28	35	22	47	26	33	24
49	8	63	10	51	6	61	12
31	42	17	40	29	44	19	38
3	54	13	60	1	56	15	58
46	27	36	21	48	25	34	23
50	7	64	9	32	5	62	11
32	41	18	39	30	43	20	37
4	53	14	59	2	55	16	57

Fig. 1.

here explained in connection with the 12th order square. The square lattice of the subsidiary square, Fig. 2, is, for convenience of construction, divided into square sections of 16 cells each. In each of the corner sections (regardless of the size of the square to be formed) are placed four 1's, their position to be as shown in Fig. 2. Each of these 1's is the initial number of the series 1, 2, 3, ..., $(n/4)^2$, which must be written in the lattice in natural order, each number falling in the same respective cell of a 16-cell section as the initial number. Two of these series are indicated in Fig. 2 by circles enclosing the numbers, and inspection will show that each of the remaining series of numbers is written in the lattice in the same manner, though they are in a reversed or reflected order. Any size subsidiary square thus filled possesses all the magic features of the final square.

A second subsidiary square of the 4th order is constructed with the series $0, (n/4)^2, 2(n/4)^2, 3(n/4)^2, \dots, 15(n/4)^2$, which must be so arranged as to produce a pandiagonal magic such as is shown in Fig. 3. It is obvious that if this square is pandiagonal, several of these squares may be contiguously arranged to form a larger

1	9	7	3	4	6	4	6	7	3	1	9
1	9	7	3	4	6	4	6	7	3	1	9
7	3	1	9	4	6	4	6	1	9	7	3
7	3	1	9	4	6	4	6	1	9	7	3
2	8	8	2	5	5	5	5	8	2	2	8
2	8	8	2	5	5	5	5	8	2	2	8
8	2	2	8	5	5	5	5	2	8	8	2
8	2	2	8	5	5	5	5	2	8	8	2
3	7	9	1	6	4	6	4	9	1	3	7
3	7	9	1	6	4	6	4	9	1	3	7
9	1	3	7	6	4	6	4	3	7	9	1
9	1	3	7	6	4	6	4	3	7	9	1

Fig. 2.

99	54	72	45
108	9	135	18
63	90	36	81
0	117	27	126

Fig. 3.

square that is pandiagonal and 4^2 -ply, and also has the concentric features previously mentioned.

Fig. 3 is now added to each section of Fig. 2, cell to cell, which will produce the final magic square in Fig. 4.

With a little practice, any size square of order $4m$ may be con-

structed without the use of subsidiary squares, by writing the numbers directly into the square and following the same order of numeral procession as shown in Fig. 5. Other processes of direct construction may be discovered by numerous arrangements and combinations of the subsidiary squares.

Fig. 5 contains pandiagonal squares of the 4th, 8th, 12th and 16th orders and is 4²-ply. The 8th and 16th order squares are also magic in their Franklin bent diagonals.

These concentric squares involve another magic feature in

100	63	79	48	103	60	76	51	106	57	73	54
109	18	142	21	112	15	139	24	115	12	136	27
70	93	37	90	67	96	40	87	64	99	43	84
7	120	28	135	4	123	31	132	1	126	34	129
101	62	80	47	104	59	77	50	107	56	74	53
110	17	143	20	113	14	140	23	116	11	137	26
71	92	38	89	68	95	41	86	65	98	44	83
8	119	29	134	5	122	32	131	2	125	35	128
102	61	81	46	105	58	78	49	108	55	75	52
111	16	144	19	114	13	141	22	117	10	138	25
72	91	39	88	69	94	42	85	66	97	45	82
9	118	30	133	6	121	33	130	3	124	36	127

Fig. 4.

respect to zig-zag strings of numbers. These strings pass from side to side, or from top to bottom, and bend at right angles after every fourth cell as indicated by the dotted line in Fig. 5. It should be noted, however, that in squares of orders $8m+4$ the central four numbers of a zig-zag string must run parallel to the side of the square, and the string must be symmetrical in respect to the center line of the square which divides the string in halves. For example in a square of the 20th order, the zig-zag string should be of this form



and not of this form



In fact any group or string of numbers in these squares, that is symmetrical to the horizontal or vertical center line of the magic and is selected in accordance with the magic properties of the 16-cell subsidiary square, will give the sum $[r(n^2+1)]/2$, where r = the number of cells in the group or string, and n = the number of cells in the edge of the magic. One of these strings is exemplified in Fig. 5 by the numbers enclosed in circles.

1	224	61	228	5	220	57	232	9	216	53	236	13	212	49	240
113	176	77	148	117	172	73	152	121	168	69	156	125	164	65	160
205	20	241	48	201	24	245	44	197	28	249	40	193	32	253	36
189	100	129	96	185	104	133	92	181	108	137	88	177	112	(14)	84
2	223	62	227	6	219	58	231	10	215	54	235	(14)	211	50	239
114	175	78	147	116	171	74	151	122	167	70	135	126	163	66	159
206	19	242	47	202	23	246	43	198	(27)	250	39	194	31	254	35
190	99	130	95	186	103	134	91	182	(107)	138	87	178	111	142	83
3	222	63	226	7	218	59	230	11	(214)	55	234	15	210	51	238
115	174	79	146	119	170	75	150	123	(166)	71	134	127	162	67	158
207	18	243	46	203	22	247	42	199	26	(25)	38	195	30	255	34
191	98	(31)	94	187	102	135	90	183	106	139	(86)	(179)	110	143	82
4	221	64	225	8	217	60	229	12	213	56	233	16	209	(52)	237
116	173	80	145	120	169	76	149	124	163	72	153	128	161	68	157
208	17	244	45	204	21	248	41	200	25	252	37	196	29	256	33
192	97	132	93	188	101	136	89	184	105	140	85	180	109	144	81

Fig. 5.

To explain what is meant above in reference to selecting the numbers in accordance with the magic properties of the 16-cell subsidiary square, note that the numbers, 27, 107, 214, 166, in the exemplified string, form a magic row in the small subsidiary square, 70, 235, 179, 30 and 251, 86, 14, 163 form magic diagonals, and 66, 159, 255, 34 and 141, 239, 82, 52 form ply groups.

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